Ahuja 9.11 sensitivity analysis for min cost flows

Goal: study changes in optimal solutions to min cost flow problem for $N=(V,A,\ell\equiv 0,u,b,c)$ when there are changes in one or more of u,b,c

Assumption: u,b,c are all integer valued

Observation: Enough to consider a change of 1 for each of them, as we can express any change as a series of changes of 1 unit

We assume that x^* is optimal and feasible in N and that π is a potential which certifies this

Thus we have

$$(*) \qquad c_{ij}^{\pi} \geq 0 \quad \forall ij \in N(x^*) \qquad \left(c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)\right)$$

and (by complementary slackness 9.8a–9.8c)

$$(9.8a) \qquad c_{ij}^{\pi} > 0 \Rightarrow x_{ij}^{*} = 0$$

$$(9.8b) \qquad 0 < x_{ij}^{*} < u_{ij} \Rightarrow c_{ij}^{\pi} = 0$$

$$(9.8c) \qquad c_{ij}^{\pi} < 0 \Rightarrow x_{ij}^{*} = u_{ij}$$

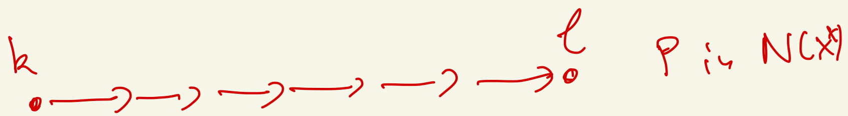
$b(k) \Leftarrow b(k)+1$ and $b(\ell) \Leftarrow b(\ell)-1$

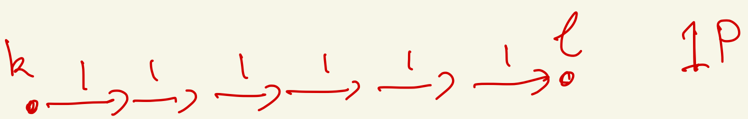
Now x^* is no longer feasible as $b_{x^*}(k) = b(k)-1$ and $b_{x^*}(\ell) = b(\ell)+1$

- x^* is still optimal (has min cost among all flows x' with $b_{x'} = b_x$)
- So by the buildup theorem
  - Either there is no feasible flow in new N (with b updated)

  or

  - We can find a new optimal feasible flow x' by
    1) Finding a shortest (k,ℓ)-path P in $N(x^*)$ wrt c^π
    2) $x' \Leftarrow x^* \oplus 1P$, where $1P$ is a path flow of value 1 along P in $N(x^*)$





$u_{pq} \leftarrow u_{pq} + 1$:

- x^* is still feasible
- If $c^{\pi}_{pq} \geq 0$ then x^* is still optimal by (*) as $c^{\pi}_{ij} \geq 0$ $\forall ij \in N(x^*)$
- suppose $c^{\pi}_{pq} < 0$

  then (*) does not hold as pq is an arc in $N(x^*)$.

  Note that $x^*_{pq} = u_{pq} - 1$ after changing u_{pq} as x^*, π satisfied (*) before we changed.

- We must have $1 \leq x^*_{pq}$ as pq is an arc of N, implying that $u_{pq} \geq 1$ before we changed
- Thus $qp \in A(N(x^*))$ so there exists a (q,p)-path in $N(x^*)$

Now we construct a new optimal and feasible flow as follows:

1. $x^*_{pq} \Leftarrow x^*_{pq} + 1$

By (*) and the fact that now $x^*_{pq} = u_{pq}$ the new x^* is optimal but it is not feasible as $b_{x^*}(p) = b(p)+1$ and $b_{x^*}(q) = b(q) - 1$

2. Let Q be a shortest (q,p)-path in $N(x^*)$ (it exists as qp is an arc in $N(x^*)$)

3. Form the flow x' by $x' \Leftarrow x^* \oplus 1Q$
Then x' is feasible in N and it is optimal by the buildup theorem

$u_{pq} \Leftarrow u_{pq} - 1$

Case 1 $\quad x^*_{pq} < u_{pq}$ before change

Then $x^*_{pq} \le u_{pq}$ after change and

(*) holds $\Rightarrow$ x^* is optimal and

feasible

Case 2 $\quad x^*_{pq} = u_{pq}$ before change

Then x^* is still optimal (as (*) holds)

But x^* is no longer feasible

1. let $x^*_{pq} \Leftarrow x^*_{pq} - 1$

   Then x^* satisfies capacities, but

   $b_{x^*}(p) = b(p) - 1$ and $b_{x^*}(q) = b(q) + 1$

2. Note that if $\exists$ feasible flow $\hat{x}$ in new N

   then $\exists$ a (p,q)-path flow $\tilde{x}$ of value 1 s.t

   $\hat{x} = x^* \oplus \tilde{x}$ in $N(x^*)$

3. If no (p,q)-path in $N(x^*)$ then there is
no solution (after changing U_{pq})

So suppose $\exists$ (p,q)-path in $N(x^*)$

4. Let R be a shortest (p,q)-path in $N(x^*)$ wrt c^{π}

Then, by the buildup theorem

$x' = x^* \oplus 1R$ is optimal and

feasible in N.

$c_{pq} \leftarrow c_{pq} + 1$

Then c^{π}_{pq} also is increased by 1.

- If $c^{\pi}_{pq} < 0$ before change then $c^{\pi}_{pq} \leq 0$ after
- if $c^{\pi}_{pq} > 0$ before change then $c^{\pi}_{pq} > 0$ after

In both cases (*) holds after change as x^*, π satisfied (*) before the change.

- If $c^{\pi}_{pq} = 0$ before change and $x^*_{pq} = 0$, then (*) still holds after change

Suppose now that $c^{\pi}_{pq} = 0$ before change and $x_{pq} > 0$
Then $qp \in N(x^*)$ and $c^{\pi}_{qp} = -c^{\pi}_{pq} < 0$ so (*) is violated.

There are 2 possible ways to find a new optimal flow:

1) Change x^*_{pq} to $x^*_{pq} \leftarrow 0$ and keep π

2) change π and possibly also x^*_{pq}

1) Let $k = x^*_{pq}$ and set $x^*_{pq} \leftarrow 0$

Now $b_{x^*}(p) = b(p) - k$ and $b_{x^*}(q) = b(q) + k$

so x^* is no longer feasible.

It is still optimal as (*) holds wrt x^*, π

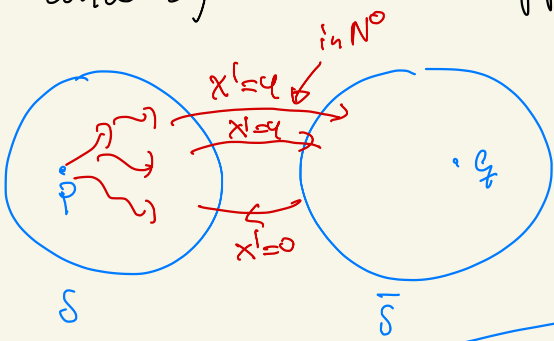
- Let N^0 be the subnetwork of $N(x^*)$ containing the arcs ij of $N(x^*)$ for which $c^{\pi}_{ij} = 0$
- Recall that if y is a (p,q)-flow of value k' in N^0, then $x' = x \oplus y$ is an optimal flow and if $k' = k$, then x' is also feasible in N and we are done.

2) Hence we may assume that the value v^0 of a maximum (p,q)-flow in N^0 satisfies $v^0 < k$

Let y be such a maxflow and use MaxFlowMinCut thm:

Set $x' = x^* \oplus y$. Then $b_{x'}(p) = b(p) - (k - v^0)$, $b_{x'}(q) = b(q) + (k - v^0)$

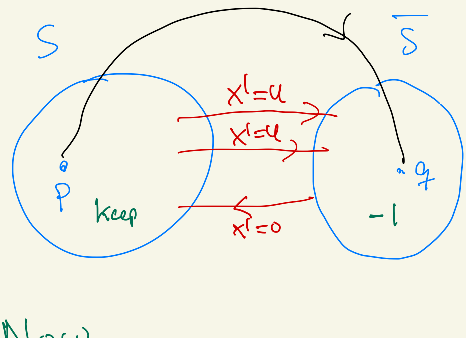
Let $(S, \bar{S})$ be a minimum (p,q)-cut in N^0 and by MFMC thm applied to y in N^0



$S = \{ v \mid \exists (p,v)\text{-path in } N^0(y) \}$

[recall that $N(x^*)(y) = N(x^* \oplus y)$] $N(x')$

$$\boxed{\text{Note that } x' \text{ is optimal wrt } \pi \text{ as we add } y \text{ which is non zero only on arcs with } c^{\pi}_{ij} = 0}$$



update π:

$$\pi'(v) = \begin{cases} \pi(v) & v \in S \\ \pi(v) - 1 & v \in \bar{S} \end{cases}$$

$$c^{\pi'}_{ij} = c_{ij} - \pi'(i) + \pi'(j)$$

Now

$$c^{\pi'}_{ij} = \begin{cases} c^{\pi}_{ij} & \text{if } i,j \in S \text{ or } i,j \in \bar{S} \\ c^{\pi}_{ij} - 1 & \text{if } i \in S,\ j \in \bar{S} \\ c^{\pi}_{ij} + 1 & \text{if } i \in \bar{S},\ j \in S \end{cases}$$

$$(\square)\quad c_{ij}^{\pi'} = \begin{cases} c_{ij}^{\pi} \text{ if } i,j \in S \text{ or } i,j \in \bar{S} \\ c_{ij}^{\pi} - 1 \text{ if } i \in S, j \in \bar{S} \\ c_{ij}^{\pi} + 1 \text{ if } i \in \bar{S}, j \in S \end{cases}$$

In particular $c_{pq}^{\pi'} = 0$

as x', π is an optimal pair we have

(9.8a) $\quad c_{ij}^{\pi} > 0 \Rightarrow x'_{ij} = 0$

(9.8b) $\quad 0 < x'_{ij} < u_{ij} \Rightarrow c_{ij}^{\pi} = 0$

(9.8c) $\quad c_{ij}^{\pi} < 0 \Rightarrow x'_{ij} = u_{ij}$

We claim that we also have (9.8a) – (9.8c) holding for x', π':

(9.8a): By $(\square)$ we only need to check arcs where $c_{ij}^{\pi} = 0$ and $c_{ij}^{\pi'} > 0$

Thus $i \in \bar{S}$ and $j \in S$ and hen

$x' = 0$ as we saw

(9.8b) suppose $0 < x'_{ij} < u_{ij}$ then

$i, j \in S$ or $i, j \in \bar{S}$

[as arcs $S \to \bar{S}$ have $x' = u$
and arcs $\bar{S} \to S$ have $x' = 0$]

and hen $c^{\pi'}_{ij} = c^{\pi}_{ij} = 0$ as

9.8b holds for x', π

(9.8c) suppose $c^{\pi'}_{ij} < 0$

if $c^{\pi}_{ij} < 0$ then $x'_{ij} = u_{ij}$

as π, x' satisfy (9.8c)

so $c^{\pi}_{ij} = 0$ and $i \in S, j \in \bar{S}$

and for these arcs we saw that $x' = u$

Conclusion x', π' is an optimal pair

Recall that $b_{x'}(p) = b(p) - (k - v^0)$

$$b_{x'}(q) = b(q) + (k - v^0)$$

We saw that x', π' is an optimal pair and that $c^{\pi'}_{pq} = 0$

Let $$x''_{ij} = \begin{cases} x'_{ij} & ij \neq pq \\ x'_{ij} + (k - v^0) & \text{if } ij = pq \end{cases}$$

Then x'' is a feasible flow in N

and it is optimal as

x'', π' is an optimal pair.