Submodularity of the degree functions of digraphs

The following simple observation plays a central role in many proofs of connectivity results.

Proposition 7

Let $D = (V, A)$ be a directed multigraph and let X, Y be subsets of V. Then the following holds:

$$\begin{aligned} d^+(X) + d^+(Y) &= d^+(X \cup Y) + d^+(X \cap Y) + d(X, Y) \\ d^-(X) + d^-(Y) &= d^-(X \cup Y) + d^-(X \cap Y) + d(X, Y). \quad (1) \end{aligned}$$

Proof: Each of these equalities can easily be proved by considering the contribution of the different kinds of arcs that are counted on at least one side of the equality.

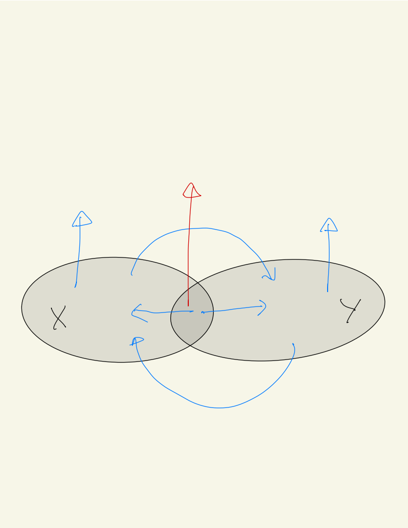


Figure: The various types of arcs that contribute to the out-degrees of the sets $X, Y, X \cap Y$ and $X \cup Y$.

A set function f on a groundset S is **submodular** if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq S$. The next corollary which follows directly from Proposition 7 is very useful, as we shall see many times in this course.

Corollary 8

For an arbitrary directed multigraph D, d_D^+, d_D^- are submodular functions on $V(D)$.

Proof of Menger's theorem via submodularity

The proof we give is due to Frank. We want to prove the following.

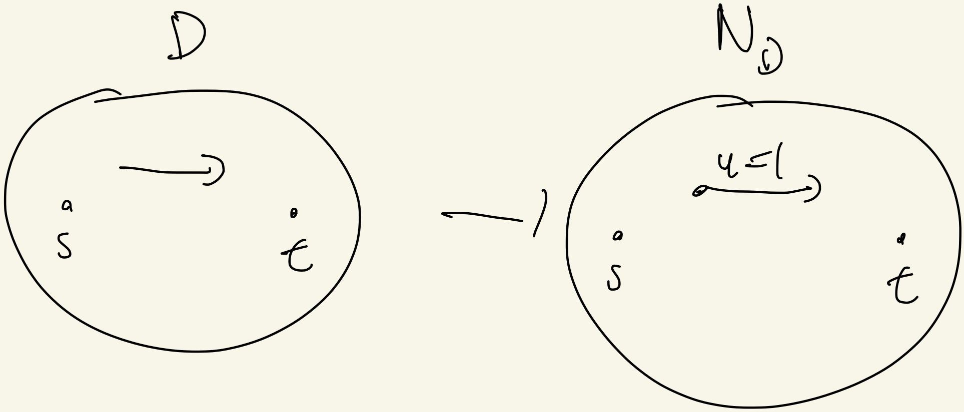
Theorem 9 (Menger 1927)

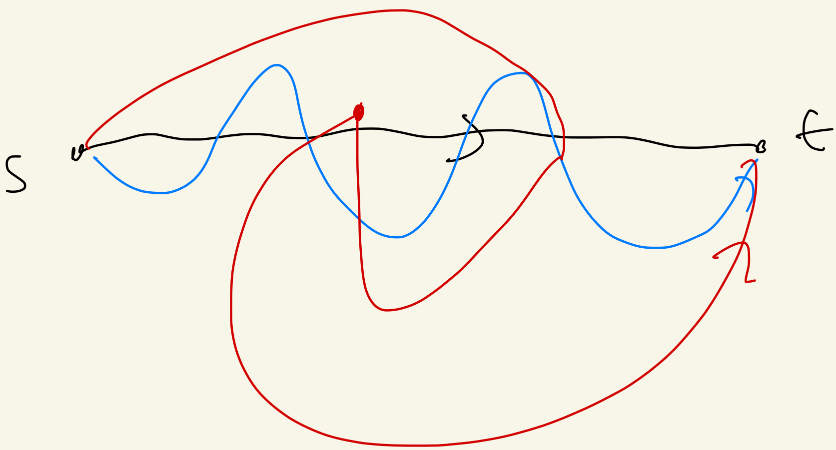
Let $D = (V, A)$ be a directed multigraph and s, t distinct vertices of V. Then the maximum number of arc-disjoint (s, t)-paths in D is equal to the minimum out-degree $d^+(X)$ of a set X which contains s but not t.

Proof: An (s, t)-**cut** is a set of arcs of the form $(X, \bar{X})$ where $s \in X$, $t \in \bar{X}$
Let k be the minimum size of an (s, t)-cut, that is, the minumum out-degree $d^+(X)$ of a set X with $s \in X, t \in \bar{X}$. Then So we have

$$d^+(X) \geq k \;\; \forall \; X \subset V - t \text{ with } s \in X \tag{2}$$

Clearly the maximum number of arc-disjoint (s, t)-paths is at most k.





max # arc-disjoint (s,t)

$= \{ |x^*| \mid x^* \text{ is an } (s,t)\text{-flow} \}$

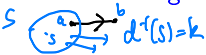
$= \min u(S, \bar{S}) \quad s \in S, t \in \bar{S}$

$= \min \{ d^+(S) \mid s \in S, t \in \bar{S} \}$

The proof of the other direction is by induction on the number of arcs in D.

- The base case is when D has precisely k arcs. Then these all go from s to t and thus D has k arc-disjoint (s, t)-paths. Hence we can proceed to the induction step.

- Call a vertex set U **tight** if $s \in U, t \notin U$ and $d^+(U) = k$. If some arc xy does not leave any tight set, then we can remove it without creating an (s, t)-cut of size $(k-1)$ and the result follows by induction. Hence we can assume that every arc in D leaves a tight set.



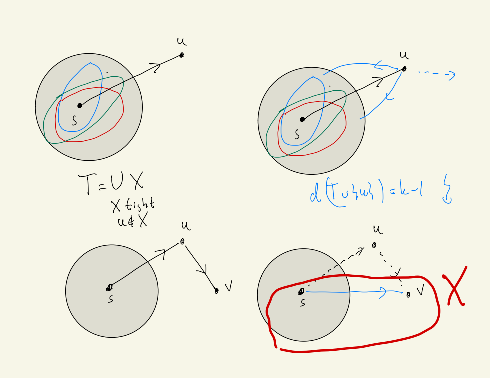
- **Claim:** If X and Y are tight sets, then so are $X \cap Y$ and $X \cup Y$. To see this we use the submodularity of d^+. First note that each of $X \cap Y$ and $X \cup Y$ contains s and none of them contains t. Hence, by (2), they both have degree at least k in D. Now using (1) we conclude

$$k + k = d^+(X) + d^+(Y) \geq d^+(X \cup Y) + d^+(X \cap Y) \geq k + k, \quad (3)$$

by the remark above. It follows that each of $X \cup Y$ and $X \cap Y$ is tight and the claim is proved.

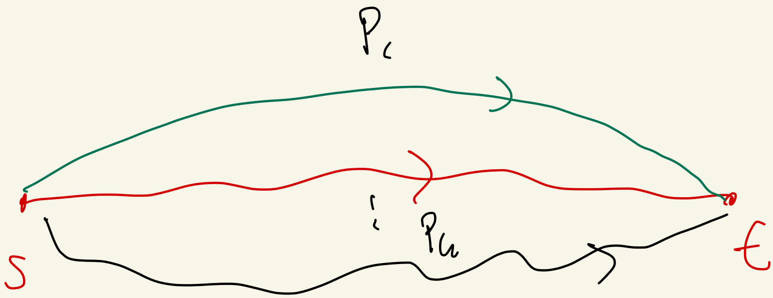
- If every arc in D is of the from st, then we are done, so we may assume that D has an arc su where $u \neq t$.

- Let T be the union of all tight sets that do not contain u. Then $T \neq \emptyset$, since the arc su leaves a tight set.
- By the claim, T is also tight.
- Now consider the set $T \cup \{u\}$.
- If there is no arc from u to $V - T$, then $d^+(T \cup \{u\}) \leq k - 1$, contradicting (2) since $T \cup \{u\}$ contains s but not t. Hence there must be some $v \in V - T - u$ such that $uv \in A(D)$.



- Now let D' be the digraph we obtain from D by replacing the two arcs su, uv by the arc sv.
- Suppose D' contains an (s,t)-cut of size less than k. That means that some set X containing s but not t has out-degree at most $k-1$ in D'.
- Since $d^+_D(X) \geq k$ it is easy to see that we must have $s, v \in X$ and $u \notin X$. Hence $d^+_D(X) = k$ and now we get a contradiction to the definition of T (since we know that $v \notin T$).
- Thus every (s,t)-cut in D' has size at least k.
- Since D' has fewer arcs than D it follows by induction that D' contains k arc-disjoint (s,t)-paths.
- At most one of these can use the new arc sv (in which case we can replace this arc by the two we deleted).
- Thus it follows that D also has k arc-disjoint (s,t)-paths.

□



k-arc-disjoint (s,t)-paths in D'

if no P_i uses s → v then

all P_i's are paths in D → done

so assume s• → v is in P_1 wlog