BJG 7.5 + 7.6 Splitting off and Increasing arc-connectivity

A digraph $D = (V, A)$ is strong if and only if it contains a directed path from x to y for every choice of vertices $x, y \in V$.
A digraph $D = (V, A)$ is ***k*-arc-strong** if $D - X$ is strong for every subset X of at most $k - 1$ arcs.

Proposition 1

Let $D = (V, A)$ be a directed multigraph and let X, Y be subsets of V. Then the following holds:

$$\begin{aligned} d^+(X) + d^+(Y) &= d^+(X \cup Y) + d^+(X \cap Y) + d(X, Y) \\ d^-(X) + d^-(Y) &= d^-(X \cup Y) + d^-(X \cap Y) + d(X, Y). (1) \end{aligned}$$

In Frank's proof of Menger's theorem, we saw how one could apply the idea of replacing two arcs incident to some vertex by one and thereby apply induction. In this talk we shall see yet another indication that this type of operation can be very useful. We consider a directed multigraph D with a special vertex s. We always assume that

$$d_D^+(s) = d_D^-(s). \tag{2}$$

To emphasize that s is a special vertex we specify D as $D = (V + s, A)$ or $D = (V + s, E \cup F)$ where F is the set of arcs with one end-vertex in s. Furthermore we will assume that the local arc-strong connectivity between every pair x, y of vertices in V is at least k.

By Menger's theorem this is equivalent to

$$d^+(U), d^-(U) \geq k \text{ for all } \emptyset \neq U \subset V. \tag{3}$$

Whenever a digraph $D = (V + s, A)$ satisfies (3) for some k we say that D is **k-arc-strong in V**.

We consider the operation of replacing a pair (us, sv) of arcs incident with s by one new arc uv. The operation of performing this replacement is called **splitting off** or just **splitting** the pair (us, sv) and the resulting directed multigraph is denoted by D_{uv}. The splitting of a pair (us, sv) is **admissible** if (3) holds in D_{uv}. If this is the case we will also say that the pair (us, sv) is an **admissible pair** (or an **admissible splitting**).

A set $\emptyset \neq X \subset V$ is **k-in-critical** (**k-out-critical**) if $d^-(X) = k$ ($d^+(X) = k$). When we do not want to specify whether X is k-in-critical or k-out-critical, we say that X is **k-critical**.

The following useful lemma is due to Frank:

Lemma 2 (Frank, 1992)

If X and Y are intersecting k-critical sets then one of the following holds:

(i) $X \cup Y$ *is k-critical,*

(ii) $Y - X$ *is k-critical and* $d(X \cap Y, V + s - (X \cup Y)) = 0$.

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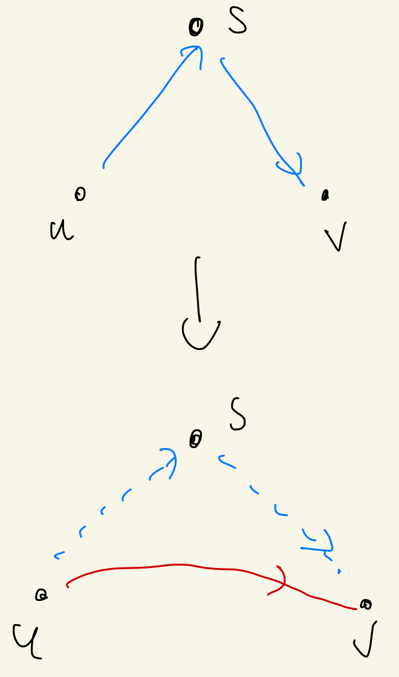
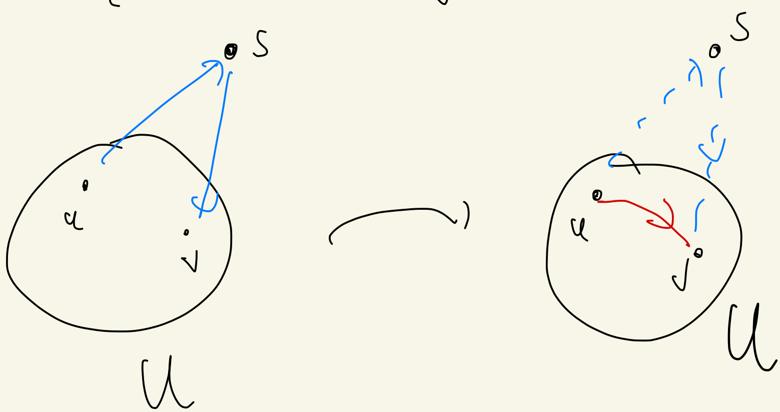
Lemma 7.5.1 in BJG



splitting off
us with sv



Intersecting sets :

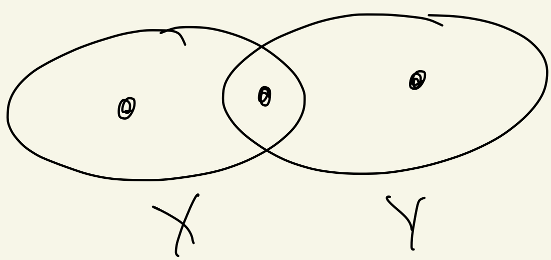
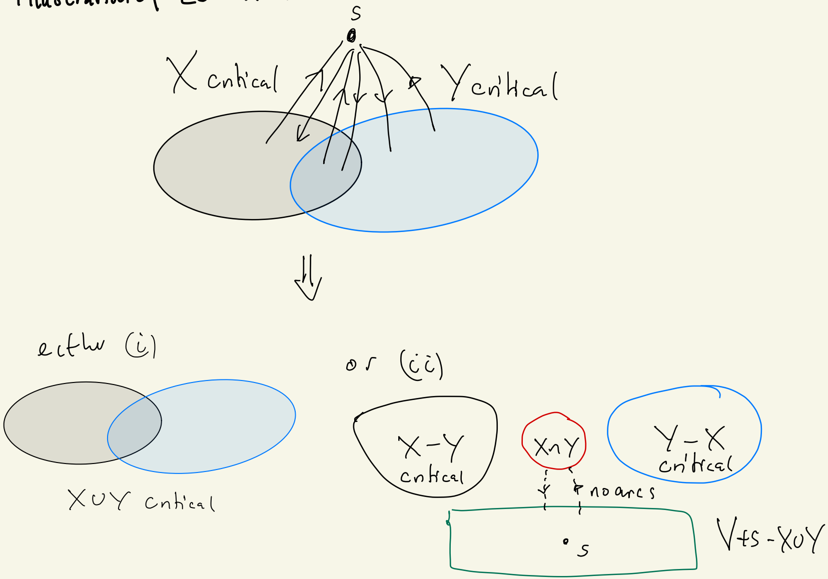


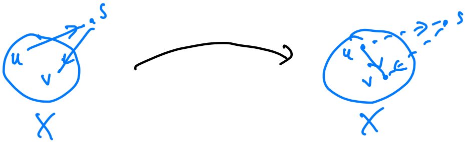
Illustration of Lemma 7.5.1



Theorem 3 (Mader, 1982)

Suppose that $D = (V + s, E \cup F)$ satisfies (3) and that $d^+(s) = d^-(s)$. Then for every arc sv there is an arc us such that the pair (us, sv) is an admissible splitting.

Proof: The proof we give is due to Frank
First note that a pair (us, sv) can be split off preserving (3) if and only if there is no k-critical set which contains both u and v.
Hence if there is no k-critical set containing v, then we are done.

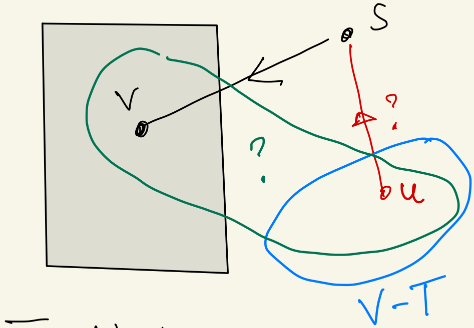


If X and Y are intersecting k-critical sets containing v, then only alternative (i) can hold in Lemma 2, because the existence of the arc sv implies that $d(V+s-(X\cup Y), X\cap Y)\geq 1$.

Hence the union T of all k-critical sets containing v is also k-critical. If we can find an in-neighbour u of s in $V-T$, then we are done, since by the choice of T, there is no k-critical set which contains u and v. So suppose that all in-neighbours of s are in T. If T is k-out-critical then

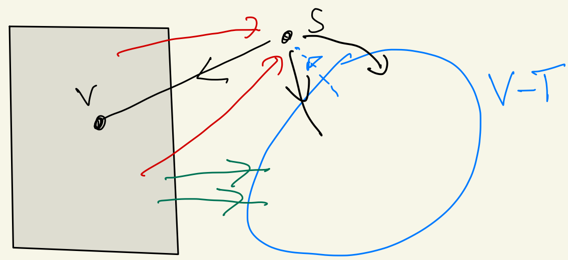
$$\begin{aligned} d^-(V-T) &= d^+(T)-d^+(T,s)+d^+(s,V-T) \\ &\leq k-(d^-(s)-d^+(s)+1) \\ &= k-1, \end{aligned}$$

since s has no in-neighbour in $V-T$ and sv is an arc from s to T (we also used $d^-(s)=d^+(s)$). This contradicts (3) so we cannot have that T is k-out-critical.



$T = \bigcup X_i$

$\{X_i \text{ critical} \wedge v \in X_i\}$



$T = \bigcup X_i$

$\{X_i \text{ critical} \wedge v \in X_i\}$

Suppose there is no arc from $V-T$ to s:
If T is out-critical then

$$
\begin{aligned}
d^-(V-T) &= d^+(T) - d^+(T,s) + d^+(s, V-T) \\
&\leq k - (d^-(s) - d^+(s) + 1) \\
&= k - 1,
\end{aligned}
$$

↯

$\Rightarrow$ T is not out-critical

So we may assume that $d^-(T) = k$

Now $d^+(V-T) = d^-(T+s)$

$$
\begin{aligned}
&= d^-(T) - d^+(s,T) + d^+(V-T,s) \\
&\leq k - 1 + 0 < k
\end{aligned}
$$

↯

Corollary 4

Suppose that $D = (V + s, E + F)$ *satisfies (3) and that* $d^+(s) = d^-(s)$*. Then there exists a pairing* $((u_1 s, s v_1), \ldots, (u_r s, s v_r))$, $r = d^-(s)$, *of the arcs entering* s *with the arcs leaving* s *such that replacing all arcs incident with* s *by the arcs* $u_1 v_1, \ldots, u_r v_r$ *and then deleting* s, *we obtain a* k*-arc-strong directed multigraph* D'. □

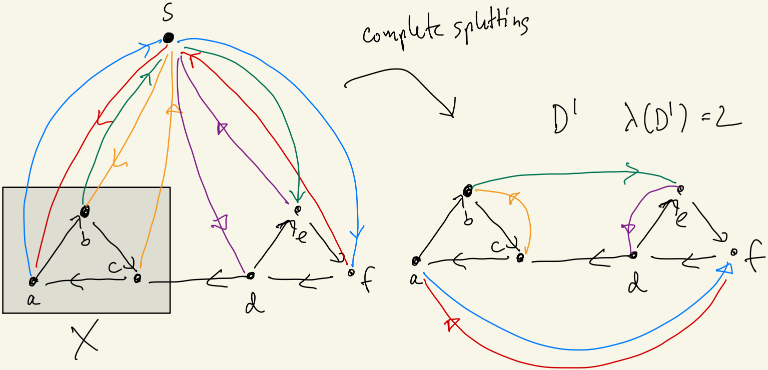
Frank and Jackson showed that for eulerian directed multigraphs one can get a stronger result. Namely, it is possible to split off all arcs incident with the special vertex s in such a way that all local arc-strong connectivities within V are preserved.

Theorem 5 (Frank, 1989, Jackson, 1988)

Let $D = (V + s, A)$ *be an eulerian directed multigraph. Then for every arc* $us \in A$ *there exists an arc* $sv \in A$ *such that* $\lambda_{D_{uv}}(x, y) = \lambda_D(x, y)$ *for all* $x, y \in V$. □

$D = (V+s, A)$

$d^+_D(U), d^-_D(U) \geq 2 \quad \forall\, U \subseteq V$



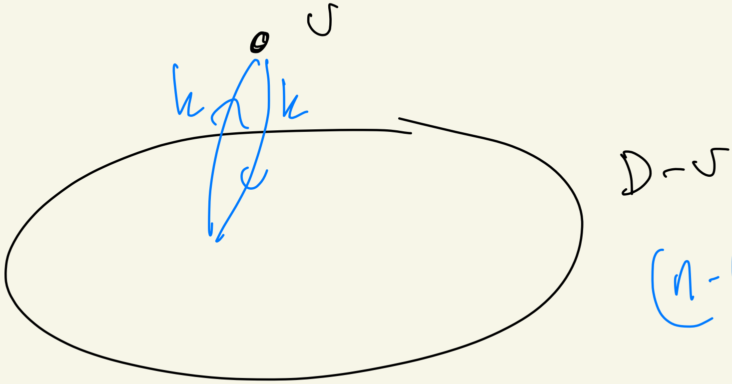
X : cannot split both (as, sb) and (cs, sa)
as $d^+(X) = 3$

increasing the arc-connectivity optimally in polynomial time

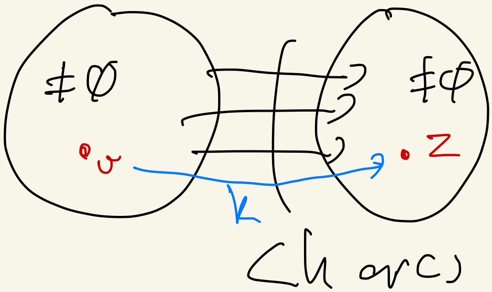
We will consider the following problem.

Arc-connectivity augmentation
Input: A directed multigraph $D = (V, A)$ and a natural number k
Question: Find a minimum cardinality set of new arcs F such that the resulting directed multigraph $D' = (V, A \cup F)$ is k-arc-strong. Remark: such a D' is called an **optimal augmentation** of D.

We will present a solution to this problem due to Frank. Frank solved the problem by supplying a min-max formula for the minimum number of new arcs as well as a polynomial algorithm to find such a minimum set of new arcs.
First let us make the simple observation that such a set F indeed exists, since we may just add k parallel arcs in both directions between a fixed vertex $v \in V$ and all other vertices in V (it is easy to see that the resulting directed multigraph will be k-arc-strong).



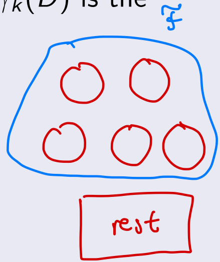
$(n-1)\cdot 2k$



Definition 6

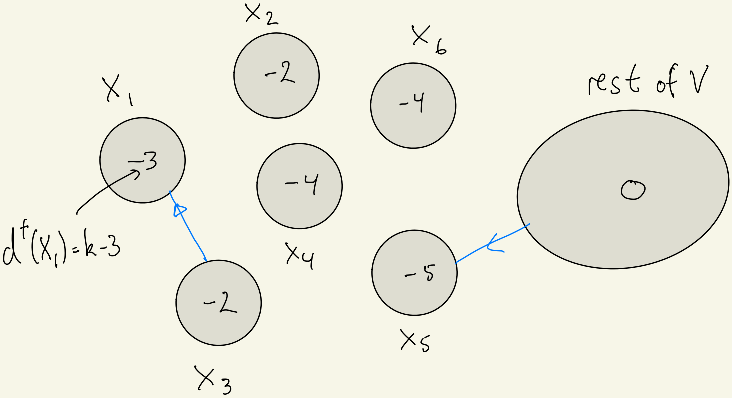
Let $D = (V, A)$ be a directed multigraph. Then $\gamma_k(D)$ is the smallest integer γ such that

$$\sum_{X_i \in \mathcal{F}} (k - d^-(X_i)) \leq \gamma \text{ and}$$
$$\sum_{X_i \in \mathcal{F}} (k - d^+(X_i)) \leq \gamma,$$



for every subpartition $\mathcal{F} = \{X_1, \ldots, X_t\}$ of V.

We call $\gamma_k(D)$ the **subpartition lower bound for arc-strong connectivity**. By Menger's theorem, D is k-arc-strong if and only if $\gamma_k(D) \leq 0$. Indeed, if D is k-arc-strong, then $d^+(X), d^-(X) \geq k$ holds for all proper subsets of V and hence we see that $\gamma_k(D) \leq 0$. Conversely, if D is not k-arc-strong, then let X be a set with $d^-(X) < k$. Take $\mathcal{F} = \{X\}$, then we see that $\gamma_k(D) \geq k - d^-(X) > 0$.



The sets $X_1 - X_6$ certify that we must add at least 20 new arcs in order to obtain a digraph which is k-arc-strong

Lemma 7 (Frank, 1982)

Let $D = (V, A)$ be a directed multigraph and let k be a positive integer such that $\gamma_k(D) > 0$. Then D can be extended to a new directed multigraph $D' = (V + s, A \cup F)$, where F consists of $\gamma_k(D)$ arcs whose head is s and $\gamma_k(D)$ arcs of whose tail is s such that (3) holds in D'.

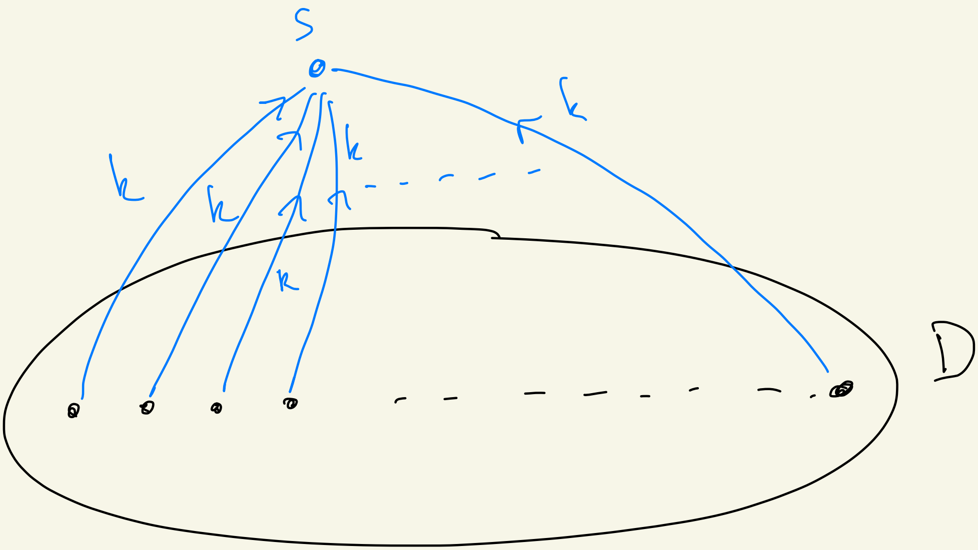
Proof: We will show that, starting from D, it is possible to add $\gamma_k(D)$ arcs from V to s so that the resulting graph satisfies

$$d^+(X) \geq k \text{ for all } \emptyset \neq X \subset V. \tag{4}$$

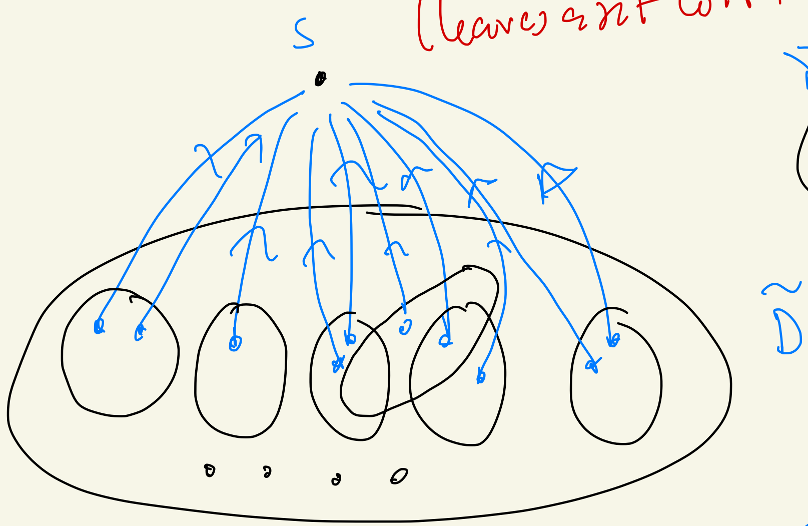
Then it will follow analogously (by considering the converse of D) that it is also possible to add $\gamma_k(D)$ new arcs from s to V so that the resulting graph satisfies

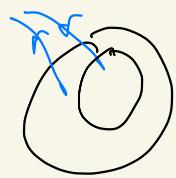
$$d^-(X) \geq k \text{ for all } \emptyset \neq X \subset V. \tag{5}$$

- First add k parallel arcs from v to s for every $v \in V$. This will certainly make the resulting directed multigraph satisfy (4).
- Now delete as many new arcs as possible until removing any further arc would result in a digraph where (4) no longer holds (that is, every remaining new arc vs leaves a k-out-critical set).
- Let $\tilde{D}$ denote the current directed multigraph after this deletion phase and let S be the set of vertices v which have an arc to s in $\tilde{D}$.
- Let $\mathcal{F} = \{X_1, \ldots, X_r\}$ be a family of k-out-critical sets such that every $v \in S$ is contained in some member X_i of $\mathcal{F}$ and assume that $\mathcal{F}$ has as few members as possible with respect to this property.
- Clearly this choice implies that either $\mathcal{F}$ is a subpartition of V, or there is a pair of intersecting sets X_i, X_j in $\mathcal{F}$.



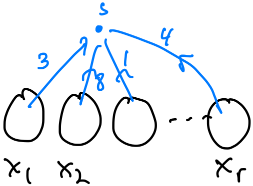
Delete blue arcs until every remaining blue arc is critical (leaves a set with out-degree k)





S = endvertices of arcs to s in $\tilde{D}$

Case 1: $\mathcal{F}$ is a subpartition of V.
Then we have



$$\begin{aligned} kr &= \sum_{i=1}^{r} d^{+}_{\tilde{D}}(X_i) \\ &= \sum_{i=1}^{r} (d^{+}_{D}(X_i) + d^{+}_{\tilde{D}}(X_i, s)) \\ &= \sum_{i=1}^{r} d^{+}_{D}(X_i) + d^{-}_{\tilde{D}}(s), \end{aligned}$$

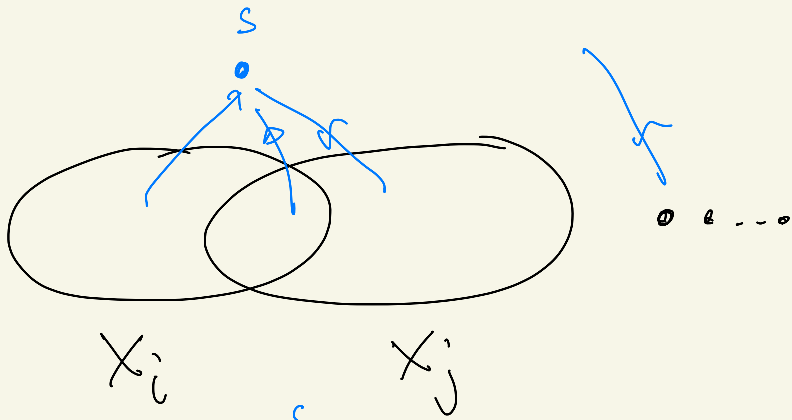
implying that $d^{-}_{\tilde{D}}(s) = \sum_{i=1}^{r}(k - d^{+}_{D}(X_i)) \leq \gamma_k(D)$, by the definition of $\gamma_k(D)$.

Case 2: Some pair $\mathbf{X_i}, \mathbf{X_j} \in \mathcal{F}$ is intersecting.
If $V - X_i \cup X_j \neq \emptyset$, then the submodularity of $d^+_{\tilde{D}}$ and (3) imply that $X_i \cup X_j$ is also k-out-critical and hence we could replace the two sets X_i, X_j by the set $X_i \cup X_j$ in $\mathcal{F}$, contradicting the choice of $\mathcal{F}$.
Hence we must have $X_i \cup X_j = V$ and $\mathcal{F} = \{X_1, X_2\}$, where without loss of generality $i = 1, j = 2$.
Let $X = V - X_1 = X_2 - X_1$ and $Y = V - X_2 = X_1 - X_2$. Then $d^-_D(X) = d^+_D(X_1)$ and $d^-_D(Y) = d^+_D(X_2)$ and hence we get

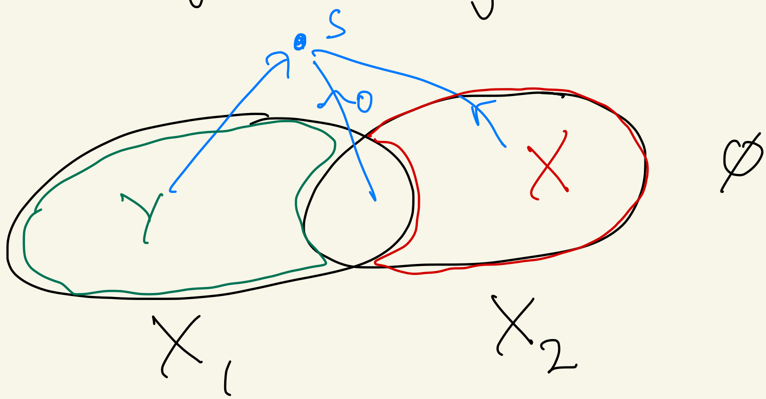
$$\begin{aligned}
\gamma_k(D) &\geq (k - d^-_D(X)) + (k - d^-_D(Y)) \\
&= k - d^+_D(X_1) + k - d^+_D(X_2) \\
&\geq k - d^+_{\tilde{D}}(X_1) + k - d^+_{\tilde{D}}(X_2) + d^-_{\tilde{D}}(s) \\
&= d^-_{\tilde{D}}(s),
\end{aligned}$$

since X_1, X_2 are k-out-critical in $\tilde{D}$. Thus $d^-_{\tilde{D}}(s) \leq \gamma_k(D)$ as claimed. $\square$

Case 2a



Case 2b



$Y \subseteq X_1 \setminus X_2$ $\quad$ $X \subseteq X_2 \setminus X_1$

$$
\begin{aligned}
\gamma_k(D) &\geq (k - d_D^-(X)) + (k - d_D^-(Y)) \\
&= k - d_D^+(X_1) + k - d_D^+(X_2) \\
&\geq k - d_{\tilde{D}}^+(X_1) + k - d_{\tilde{D}}^+(X_2) + d_{\tilde{D}}^-(s) \\
&= d_{\tilde{D}}^-(s),
\end{aligned}
$$

since X_1, X_2 are k-out-critical in $\tilde{D}$. Thus $d_{\tilde{D}}^-(s) \leq \gamma_k(D)$ as claimed.

Theorem 8 (Frank, 1992)

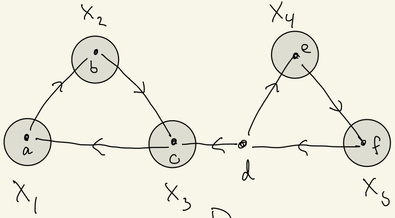
Let $D = (V, A)$ be a digraph and k a natural number such that $\gamma_k(D) > 0$. The minimum number of new arcs that must be added to D in order to give a k-arc-strong digraph $D' = (V, A \cup F)$ equals $\gamma_k(D)$.

Proof: To see that we must use at least $\gamma_k(D)$ arcs, it suffices to observe that if X and Y are disjoint sets then no new arc can increase the out-degree (in-degree) of both sets. Hence a subpartition $\mathcal{F}$ realizing the value of γ_k in Definition 6 is a certificate that we must use at least $\gamma_k(D)$ new arcs.

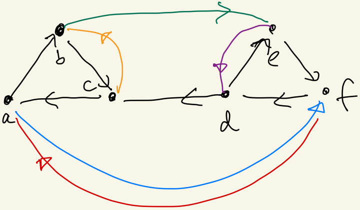
- To prove the other direction we use Mader's splitting theorem and Lemma 7.
- According to this lemma we can extend D to a new digraph $\tilde{D}$ by adding a new vertex s and $\gamma_k(D)$ arcs from V to s and from s to V, such that $d^+_{\tilde{D}}(X), d^-_{\tilde{D}}(X) \geq k \quad \forall \emptyset \neq X \subset V$
- Note that we may not need $\gamma_k(D)$ arcs in both directions, but we will need it in one of the directions by our remark in the beginning of the proof. In the case where fewer arcs are needed, say from V to s we add arbitrary arcs from V to s so that the resulting number becomes $\gamma_k(D)$.
- Now it follows from Corollary 4 that all arcs incident with s can be split off without violating (3).
- This means that, if we remove s, then the resulting graph D' is k-arc-strong.

□

$k = 2$



D



D'

$$\sum_{i=1}^{5} \left| k - d^+(X_i) \right| = 5$$

5 new arcs added and $\lambda(D') = 2$

- In the proof of Lemma 7, we never used exactly how we obtained the minimal set of arcs from V to s so that (4) held.
- The proof is valid for every such set of arcs that is minimal with respect to deletion of arcs.
- This means in particular that we can use a greedy approach to find such a set of arcs starting from the configuration with k parallel arcs from every vertex $v \in V$ to s.
- This gives rise to the following algorithm, by Frank, for augmenting the arc-strong connectivity optimally to k for any digraph D which is not already k-arc-strong:

Frank's arc-strong connectivity augmentation algorithm
Input: A directed multigraph $D = (V, A)$ and a natural number k such that $\gamma_k(D) > 0$.
Output: A k-arc-strong optimal augmentation D^* of D.

1. Let $v_1, v_2 \ldots, v_n$ be a fixed ordering of V and let s be a new vertex.
2. Add k parallel arcs from v_i to s and from s to v_i for each $i = 1, 2, \ldots, n$.
3. Starting from $i := 1$, remove as many arcs from v_i to s as possible without violating (4); If $i < n$ then let $i := i + 1$ and repeat this step;
   Let γ^- denote the number of remaining arcs from V to s in the resulting digraph.
4. Starting from $i := 1$, remove as many arcs from s to v_i as possible without violating (5); If $i < n$ then $i := i + 1$ and repeat this step;
   Let γ^+ denote the number of remaining arcs from s to V in the resulting digraph.

5. Let $\gamma = \max\{\gamma^-, \gamma^+\}$. If $\gamma^- < \gamma^+$, then add $\gamma^+ - \gamma^-$ arcs from v_1 to s; If $\gamma^+ < \gamma^-$, then add $\gamma^- - \gamma^+$ arcs from s to v_1.
6. Let D' denote the current digraph. In D' we have $d^-_{D'}(s) = d^+_{D'}(s)$ and (3) holds. Split off all arcs incident with s in D' by applying Theorem 3 γ times. Let D^* denote the resulting directed multigraph.
7. Return D^*.

Using flows this algorithm can be implemented as a polynomial algorithm for augmenting the arc-strong connectivity of a given digraph. See Exercises 7.28 and 7.30.

If we assign costs on the possible new arcs and ask for a minimum cost (rather than just minimum cardinality) set of new arcs to add to D in order to obtain a k-arc-strong digraph D', then we have the MINIMUM COST ARC-STRONG CONNECTIVITY AUGMENTATION PROBLEM.

Theorem 9

The MINIMUM COST ARC-STRONG CONNECTIVITY AUGMENTATION PROBLEM *is* $\mathcal{NP}$*-hard.*

Proof: We show that the NP-complete problem of deciding whether a directed graph has a hamiltonian cycle can be reduced to the weighted arc-strong connectivity augmentation problem in polynomial time.
Let $D = (V, A)$ be a digraph on n vertices $V = \{1, 2, \ldots, n\}$. Define weights $c(ij)$ on the arcs of the complete digraph $\overset{\leftrightarrow}{K}_n$ with vertex set V as follows:

$$c(ij) = \begin{cases} 1 & \text{if } ij \in A \\ 2 & \text{if } ij \notin A. \end{cases} \tag{6}$$

- Let $D_0 = (V, \emptyset)$ (that is, the digraph on V with no arcs).
- Since every vertex of a strong digraph is the tail of at least one arc, we need at least n arcs to make D_0 strong.
- Now it is easy to see that D_0 can be made strongly connected using arcs with total weight at most n if and only if D has a Hamilton cycle.
- Thus we have reduced the Hamilton cycle problem to the weighted arc-strong connectivity augmentation problem. Clearly our reduction can be carried out in polynomial time.

$\square$