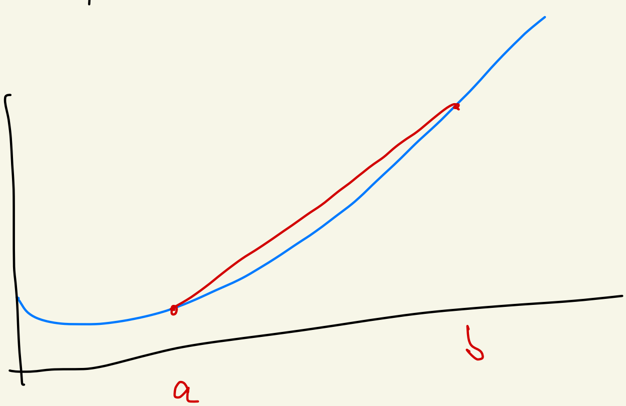
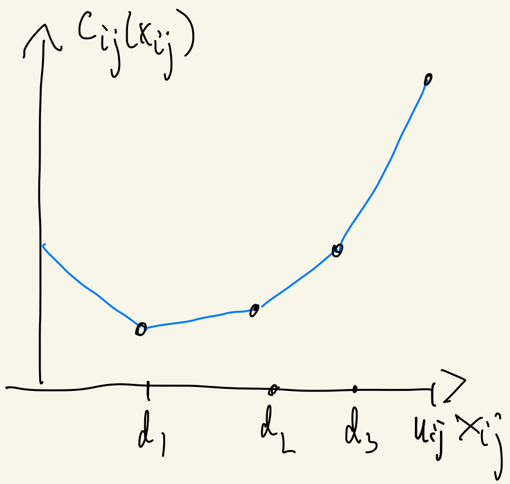
- Standard cost function in min cost flow problem c_{ij} is a constant so cost of sending x_{ij} units along ij is $c_{ij}x_{ij}$

- A function f is convex if $\forall a<b$ in the domain of f the values of f in the interval $[a,b]$ lie below the straight line from $(a, f(a))$ to $(b, f(b))$

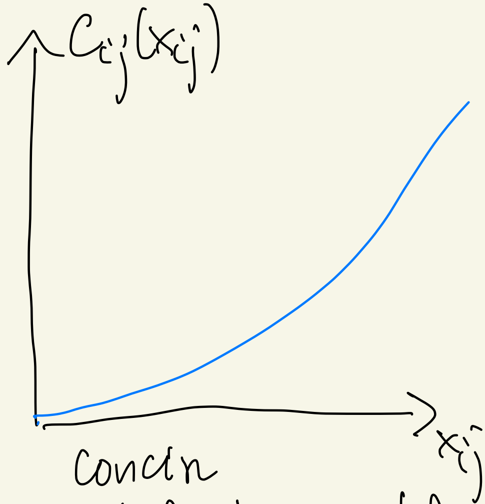


Ahuja chapter 14 convex cost flows



discrete piecewise linear

O(#breakpoints) space



convex

cost function specified in functional form

e.g. x_{ij}^4 O(1) space

Min convex cost flow:

$$\min \sum c_{ij}(x_{ij})$$

$$\text{s.t.} \quad b_x(i) = b(i) \quad \forall i \in V \qquad (*)$$

$$0 \le x_{ij} \le u_{ij} \quad \forall ij \in A$$

$$x_{ij} \in \mathbb{Z} \quad \forall ij \in A$$

consider only integer flows!

assumptions on $N=(V,A,l,u,b,c(\cdot))$

(1) $D=(V,A)$ is a digraph

(2) $b \in \mathbb{Z}^n$ (integer balances)

(3) $l_{ij}=0 \ \forall ij$

(4) $\exists (i,j)$-path of ∞ capacity $\forall i,j \in V$

(5) $c_{ij}(0)=0 \quad \forall ij \in A$

Section 14.2 on applications $\Rightarrow$ self study

14.3 Transformation to a normal min cost flow problem when c_{ij} is convex and piecewise linear for all arcs

Drawback of solution: The new network may be much larger than N as # arcs will depend on # breakpoints of the cost functions.

Assumption: each c_{ij} has exactly p breakpoints (consists of p linear segments)

so c_{ij} is constant in each of the segments

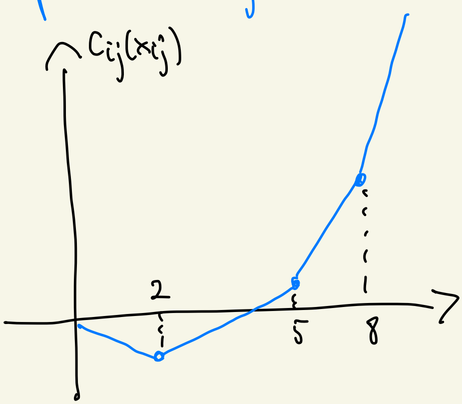
$$[d_{ij}^0, d_{ij}^1], [d_{ij}^1, d_{ij}^2] \;\cdots\cdots\; [d_{ij}^{p-1}, d_{ij}^p]$$

c_{ij}^1 $\quad$ c_{ij}^2 $\quad$ c_{ij}^p

when $d_{ij}^0 = 0$ and $d_{ij}^p = u_{ij}$

Note that p is finite as u_{ij} is finite

look at x_{ij} and split in p parts which represent the part of x_{ij} belonging to each of the p segments:



fill up from below:
in example, if $x_{ij} = 6$
then segments 1 and 2
are filled and the flow in
segment 3 is 1

Given x_{ij} we define segment flows y_{ij}^k for $k \in [p]$:

$$(\square) \quad y_{ij}^k = \begin{cases} 0 & \text{if } x_{ij} \le d_{ij}^{k-1} \\ x_{ij} - d_{ij}^{k-1} & \text{if } x_{ij} \in [d_{ij}^{k-1}, d_{ij}^k] \\ d_{ij}^k - d_{ij}^{k-1} & \text{if } x_{ij} \ge d_{ij}^k \end{cases}$$

Then

$$x_{ij} = \sum_{k=1}^{p} y_{ij}^k \quad \text{and} \quad c_{ij}(x_{ij}) = \sum_{k=1}^{p} c_{ij}^k y_{ij}^k$$

This transforms the model $(*)$ into $(**)$:

$$\min \sum_{ij \in A} \sum_{k=1}^{p} c_{ij}^k y_{ij}^k$$

$$\sum_{ij} \sum_{k=1}^{p} y_{ij}^k - \sum_{ji} \sum_{k=1}^{p} y_{ji}^k = b(i) \quad (**)$$

$$0 \le y_{ij}^k \le d_{ij}^k - d_{ij}^{k-1} \qquad \forall ij$$

$$\min \sum_{ij\in A} \sum_{k=1}^{p} c_{ij}^k y_{ij}^k$$

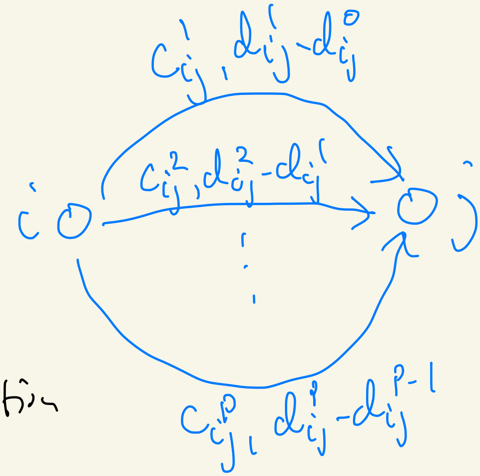
$$\sum_{ij} \sum_{k=1}^{p} y_{ij}^k - \sum_{ji} \sum_{k=1}^{p} y_{ji}^k = b(i) \quad (**)$$

$$0 \le y_{ij}^k \le d_{ij}^k - d_{ij}^{k-1} \qquad \forall\, ij$$

Claim: (**) is a min cost flow problem in the network N' when we replace

$$i \xrightarrow{\;c_{ij}(\,),\, u_{ij}\;} j$$

by



Def y is a contiguous solution to (*) if $y_{ij}^q > 0$

$$\Downarrow$$

$$y_{ij}^r = d_{ij}^r - d_{ij}^{r-1} \quad \text{for all } r < q$$

Show equivalence between (*) and (**)

Suppose x solves (*)

Def $y^1_{ij}, \dots, y^p_{ij}$ from x_{ij} as in (□)

that is, fill up the intervals $[d^0_{ij}, d^1_{ij}] \dots [d^{p-1}_{ij}, d^p_{ij}]$

in that order.

Then y is a contiguous solution and

$$c_{ij}(x_{ij}) = \sum_{k=1}^{p} c^k_{ij} y^k_{ij}$$

so the cost of x and y are the same

Suppose y is a contiguous solution to (**)

Define x_{ij} as $x_{ij} = \sum_{k=1}^{p} y^k_{ij}$ then

x is a solution to (*) and its cost

is the same as the cost of y.

Note that there may be non contiguous solutions to (**) but they are not optimal:

Suppose $y_{ij}^{\ell} = M > 0$ but $y_{ij}^{k} < d_{ij}^{k} - d_{ij}^{k-1}$ some $k < \ell$.

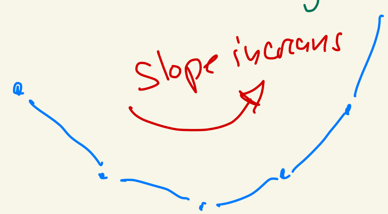
Then we get a cheaper solution to (**) by setting

$$y_{ij}^{k} \leftarrow y_{ij}^{k} + Q \quad \text{and}$$

$$y_{ij}^{\ell} \leftarrow y_{ij}^{\ell} - Q$$

where $Q = \min\{M, (d_{ij}^{k} - d_{ij}^{k-1}) - y_{ij}^{k}\}$

why? because $c_{ij}^{k} < c_{ij}^{\ell}$ by convexity of $C_{ij}()$



Conclusion: All optimal solutions to (**) are contiguous.

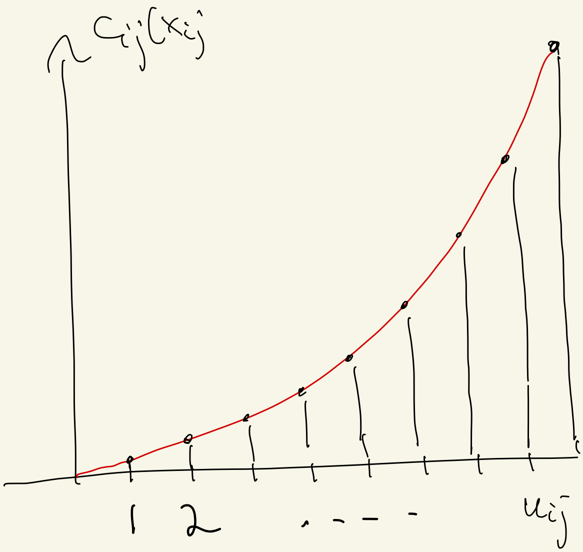
- Algorithmic challenge with reduction:

Need p copies of each original arc

- Not a problem if $c_{ij}(x_{ij})$ is specified as a piecewise linear function, since then we need (up to) p breakpoints for each arc ij.

- If instead $c_{ij}(x_{ij})$ is in functional form (e.g. $c_{ij}(x_{ij}) = x_{ij}^2$)

and we represent $c_{ij}(x_{ij})$ by a piecewise linear approximation with u_{ij} lines then we may need much more space in model (✗✗)



14.4 pseudopolynomial algorithms

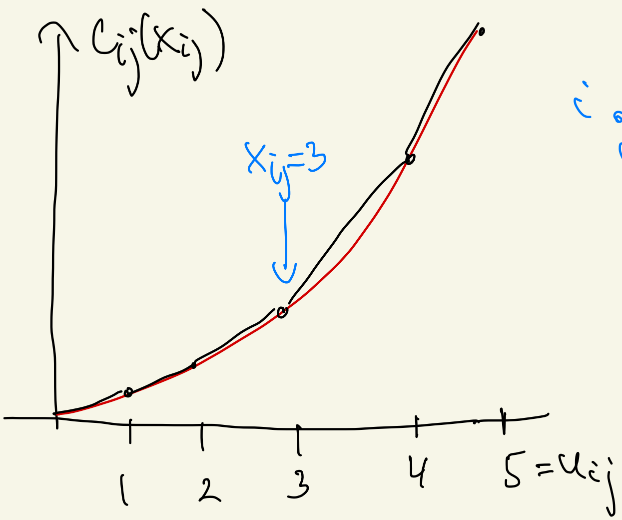
Modification of Cycle Cancelling and Buildup method for standard min cost flow.

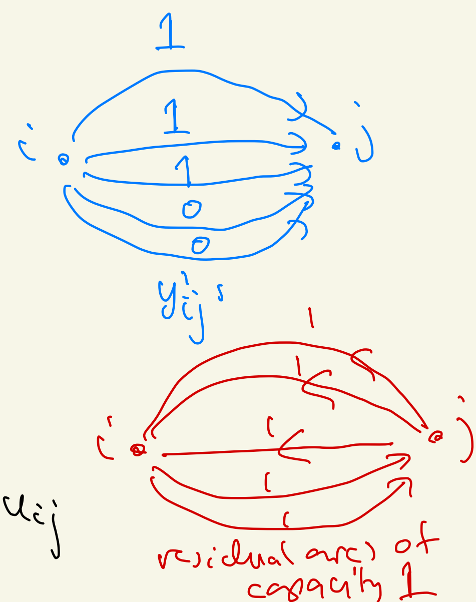
New approach:

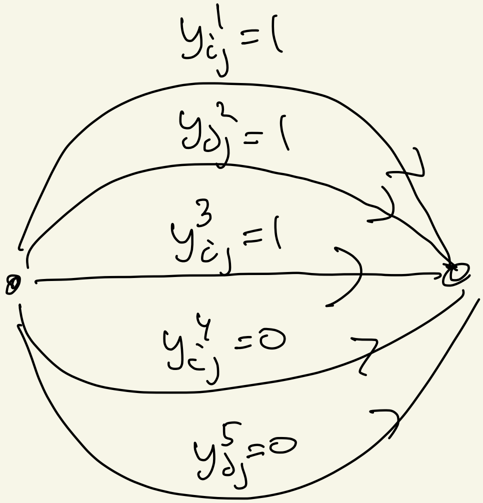
Do not make all p copies of an arc ij but introduce only the relevant copies.

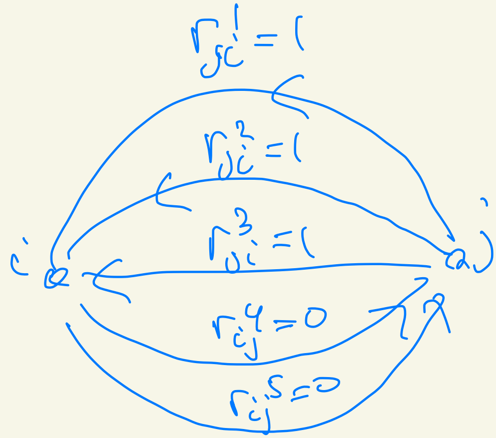
The residual network has (potentially) many arcs in both directions between i and j

Example based on figure 14.2:









Effect of increasing $x_{ij} \to x_{ij}+1$:

$y_{ij}^4 \leftarrow 1$ as $c_{ij}^4 < c_{ij}^5$

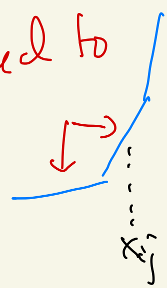
Effect of decreasing $x_{ij} \to x_{ij}-1$:

$y_{ij}^3 \leftarrow 0$ as $c_{ij}^3 > c_{ij}^2 > c_{ij}^1$

so $-c_{ij}^3 < -c_{ij}^2 < -c_{ij}^1$

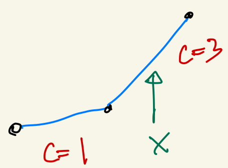
<u>Conclusion:</u> Enough to keep ij^3 and ij^4 in the residual network

This holds in general: we just need to store two arcs in $N(x)$ per arc ij



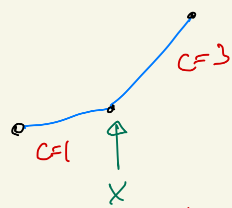
Construction of $N(x)$ $\forall ij \in A$:

- if $x_{ij} < u_{ij}$ then ij is an arc in $N(x)$ with cost $C_{ij}(x_{ij}+1) - C_{ij}(x_{ij})$
- if $0 < x_{ij}$ then ji is in $N(x)$ with cost $C_{ij}(x_{ij}-1) - C_{ij}(x_{ij})$



cost of ij in $N(x)$ is 3

cost of ji in $N(x)$ is -3



cost of ij in $N(x)$ is 3

cost of ji in $N(x)$ is -1

r_{ij} = distance from x_{ij} to next breakpoint

r_{ji} = distance from previous breakpoint to x_{ij}

Cycle cancelling for convex cost flows

Same as normal, except for construction and maintenance of $N(x)$:

Given feasible flow x

- construct $N(x)$
- while $\exists\, W \in N(x) : c(W) < 0$
  - let $\delta(W)$ be capacity of W (defined below)
  - $x \leftarrow x \oplus \delta(W) W$
  - construct $N(x)$

$\delta(W)$ is now the minimum residual capacity of an arc $ij^{\pm}$ on W

So we only increase/decrease flow in ij to the next breakpoint

Note that, by the equivalence of models (*) and (**) and the definition of $N(x)$ x is optimal if and only if $N(x)$ has no negative cycle.

Possible improvement

- After augmenting by $\delta(W)$ units along W the flow on one or more arcs ij of W has reached breakpoint so it is possible that $c(W)$ remains negative in the new residual network.

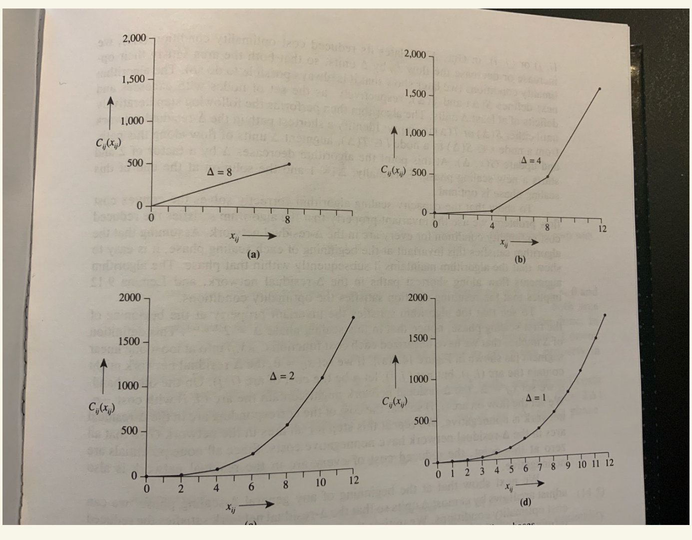
In that case we can augment along W with $\delta(W)$ units for the new value $\delta(W)$.

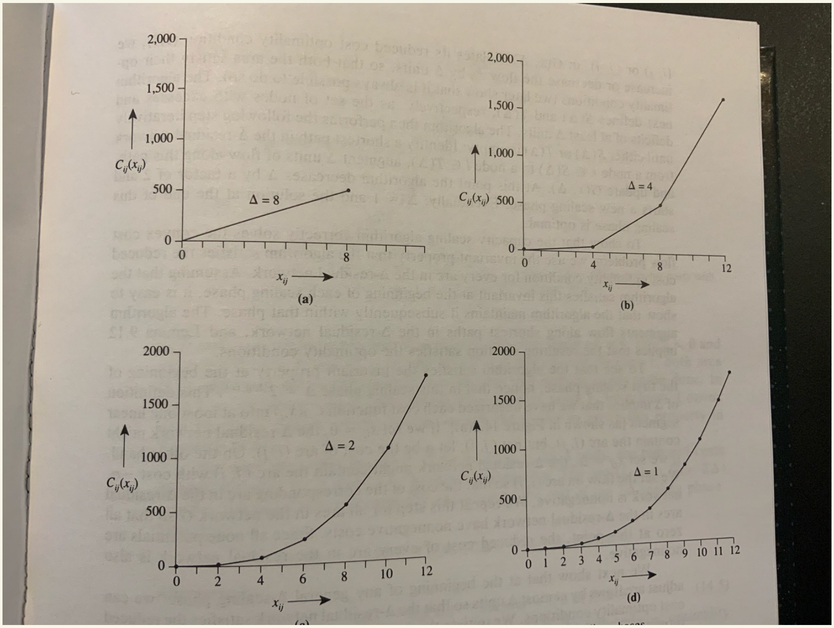
Buildup algorithm:

Again very similar except for det. of $N(x)$.
(self study!)

14.5 Obtaining a polynomial algorithm.

- Based on the capacity scaling algorithm in Section 10.2.
- The algorithm does not linearize $C_{ij}(x_{ij})$ in one step but in a logarithmic # of iterations in which the approximation becomes more and more accurate





$C_{ij}(x_{ij}) = x_{ij}^4$

$u_{ij} = 12 \in [2^3, 2^4-1) \Rightarrow \lfloor \log_2 u_{ij} \rfloor = 3 \quad \Delta = 2^3 = 8$

- step 1 one segment of length 8 and slope $\frac{8^4}{8} = 8^3$
- step 2 segment above divided into 2 segments of length 4 with slopes $\frac{4^4}{4} = 4^3$ and $\frac{8^4 - 4^4}{4}$ respectively
- In step 4 all segments have length 1

In the Δ-scaling phase we maintain the Δ-residual network $N(x,\Delta)$

- If $x_{ij}+\Delta \le u_{ij}$ then $ij \in N(x,\Delta)$ and has cost $\dfrac{c_{ij}(x_{ij}+\Delta)-c_{ij}(x_{ij})}{\Delta}$

- if $x_{ij} \ge \Delta$ then $ji \in N(x,\Delta)$ and has cost $\dfrac{c_{ij}(x_{ij}-\Delta)-c_{ij}(x_{ij})}{\Delta}$

Let $U = \max_{ij} u_{ij}$

initially $\Delta = 2^{\lfloor \log_2 U \rfloor}$ $x \equiv 0$ and $\pi \equiv 0$

x, π show that x is optimal

as $c_{ij}(0)=0$ $\forall ij$ by assumption phase Δ:

1. Construct $N(x,\Delta)$
2. $\forall ij \in A$: if ij or ji is in $N(x,\Delta)$ and has negative reduced cost we increase or decrease the flow by Δ units
3. $S(\Delta) \leftarrow \{ i \mid b(i) - b_x(i) \geq \Delta \}$
   $T(\Delta) \leftarrow \{ i \mid b_x(i) - b(i) \geq \Delta \}$
4. Find shortest path distances $d(\ )$ from some $k \in S(\Delta)$ and let P be a shortest (k,l)-path for some $l \in T(\Delta)$
5. $\pi \leftarrow \pi - d$
6. Augment x by sending Δ units along P
7. update $S(\Delta), T(\Delta)$ and if both are nonempty go to 4
8. If $\Delta > 1$ set $\Delta \leftarrow \frac{\Delta}{2}$ and goto 1

Note that in the 1-scaling phase the discretization is completed. We now prove that the final x is optimal.

$\boxed{\text{Invariant} \quad c_{ij}^{\pi} \geq 0 \text{ for every arc } ij \in N(x,\Delta)}$

$\boxed{\text{Invariant} \quad c_{ij}^{\pi} \geq 0 \text{ for every arc } ij \in N(x, \Delta)}$

Note that it is enough to show if the invariant holds initially and when the 2Δ-phase finishes. Then step 2 (changing x_{ij} by $+/- \Delta$ if $c_{ij}^{\pi} < 0$ or $c_{ji}^{\pi} < 0$ for arcs in residual network) will establish invariant and steps 4, 5 maintains the invariant

Invariant at the beginning of $\Delta = 2^{\lfloor \log_2 U \rfloor}$ phase:

- $C_{ij}(x_{ij}')$ is linearized into at most one segment (0 segments if $u_{ij} < \Delta$)

- let $\alpha = c_{ij}^{1}$ (slope of first section as discretization)
  - If we set $x_{ij} \leftarrow 0$ then $ij \in N(x, \Delta)$ with cost α and ji is not in $N(x, \Delta)$
  - if we set $x_{ij} \leftarrow \Delta$ then $ji \in N(x, \Delta)$ with cost $-\alpha$ and ij is not in $N(x, \Delta)$

- The algorithm assigns x_{ij} value 0 or Δ s.t. the resulting arc in $N(x)$ has cost ≥ 0

- We have $\pi \equiv 0$ initially so all arcs of $N(x,\Delta)$ have reduced cost ≥ 0 when the first phase starts.

So the invariant holds.

<u>General Δ-phase:</u> assuming invariant holds when the 2Δ-phase finishes.

Possible problem: in the 2Δ-phase $C_{ij}(x_{ij})$ is linearized into segments of length 2Δ and in the Δ-phase into segments of length Δ $\Rightarrow$ arc costs change in $N(x,\Delta)$: so we risk having arcs with negative cost

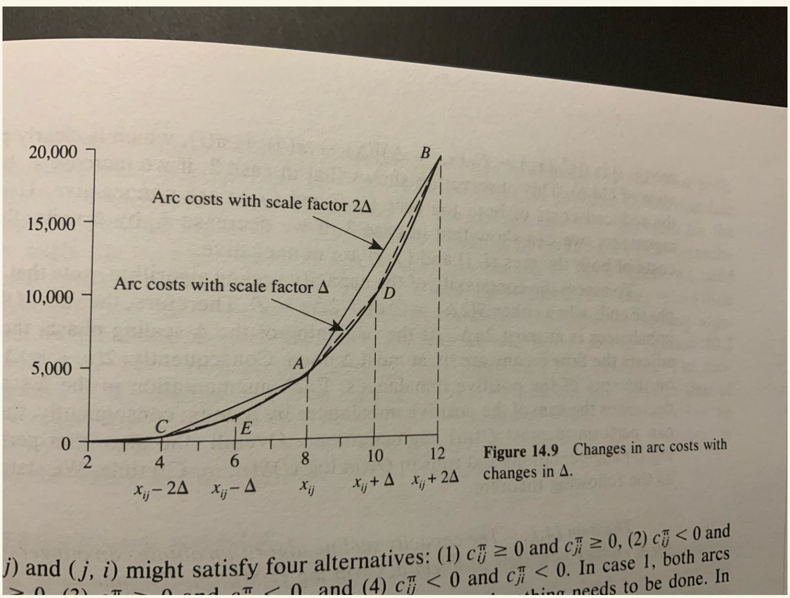


Figure 14.9 Changes in arc costs with changes in Δ.

In $N(x,2\Delta)$:

ij cost = slope of AB

ji cost = $-$slope of AC

In $N(x,\Delta)$:

ij cost = slope of AD

ji cost = $-$slope of AE

so cost ij $\searrow$ and cost ji $\nearrow$

Recall: in Δ-phase the reduced cost of ij is

$$c_{ij}^{\pi} = (c_{ij}(x_{ij}+\Delta) - c_{ij}(x_{ij}))/\Delta - \pi(i) + \pi(j)$$

There are 4 possible combinations of the signs of c_{ij}^{π} and c_{ji}^{π} in $N(x,\Delta)$ when $ij, ji \in N(x)$

(1)	(2)	(3)	(4)
$c_{ij}^{\pi} \geq 0, c_{ji}^{\pi} \geq 0$,	$c_{ij}^{\pi} < 0, c_{ji}^{\pi} \geq 0$,	$c_{ij}^{\pi} \geq 0, c_{ji}^{\pi} < 0$	, $c_{ij}^{\pi} < 0, c_{ji}^{\pi} < 0$

Case (1): nothing needed for the arc ij

Case (2): $c_{ij}^{\pi} < 0$ and $c_{ji}^{\pi} \geq 0$

We want to increase x_{ij} by Δ

$$\left[C_{ij}(x_{ij}+2\Delta) - C_{ij}(x_{ij})\right]/2\Delta - \pi(i) + \pi(j) \geq 0 \quad (2\Delta\text{-phase})$$

$$\Updownarrow$$

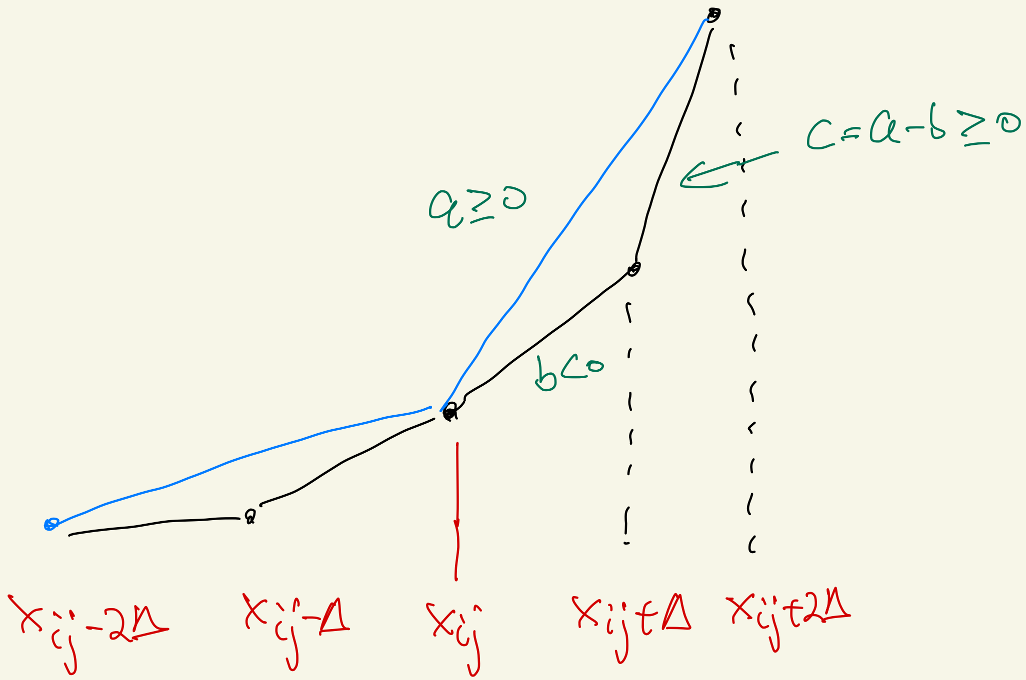
$$C_{ij}(x_{ij}+2\Delta) - C_{ij}(x_{ij}) - 2\Delta\pi(i) + 2\Delta\pi(j) \geq 0 \quad (\boxtimes)$$

$$C_{ij}(x_{ij}+2\Delta) - C_{ij}(x_{ij}+\Delta) - \Delta\pi(i) + \Delta\pi(j)$$

$$= \left[C_{ij}(x_{ij}+2\Delta) - C_{ij}(x_{ij}) - 2\Delta\pi(i) + 2\Delta\pi(j)\right]$$

$$- \left[C_{ij}(x_{ij}+\Delta) - C_{ij}(x_{ij}) - \Delta\pi(i) + \Delta\pi(j)\right]$$

$$\geq 0 \text{ as } \underset{\alpha}{\underset{\shortparallel}{[\ \ ]}} \geq 0 \text{ and } \underset{b}{\underset{\shortparallel}{[\ \ ]}} < 0 \text{ since } c_{ij}^{\pi} < 0$$



after increasing x_{ij} to $x_{ij}+\Delta$ the arc ji is in $N(x)$ which is ok since

$$c^{\pi}_{ji} = \left[c_{ij}(x_{ij}) - c_{ij}(x_{ij}+\Delta)\right]/\Delta - \pi(j) + \pi(i)$$

$$= -\left[\left(c_{ij}(x_{ij}+\Delta) - c_{ij}(x_{ij})\right)/\Delta - \pi(i) + \pi(j)\right]$$

$$= -c^{\pi}_{ij} > 0 \quad \text{as } c^{\pi}_{ij} < 0$$

<u>case 3</u> $c_{ij}^{\pi} \geq 0$, $c_{ji}^{\pi} < 0$

very similar calculations:
we decrease x_{ij} by Δ to obtain
that both ij and ji have
nonnegative reduced cost
(ji may disappear if we set $x_{ij} = 0$.)

<u>case 4</u> $c_{ij}^{\pi} < 0$, $c_{ji}^{\pi} < 0$

impossible since $c_{ij}(x_{ij})$ is convex

see exercise 14.17

Complexity (similar analysis as for sec 10.2)

- When 2Δ-phase terminates we have

$$E = \sum_{b(i) > b_x(i)} \left(b(i) - b_x(i)\right) \le 2n\Delta$$

as $S(\Delta) = \phi$
or $T(\Delta) = \phi$

- modifying x_{ij}'s at the beginning of the phase changes E by at most $2m\Delta$
  so $E \le 2(n+m)\Delta$ when we start augmenting by exactly Δ-units along shortest paths
  $\Downarrow$
  $O(m)$ augmentations in Δ-phase
- $O(\log U)$ phases
- When the $2^{\lfloor \log_2 U \rfloor}$ phase begins, we have $\sum b(i) \le mU \le 2m\, 2^{\lfloor \log_2 U \rfloor} = 2m\Delta$

$\Rightarrow$ Complexity is $O(m \log_2 U \cdot \text{Dijkstra})$