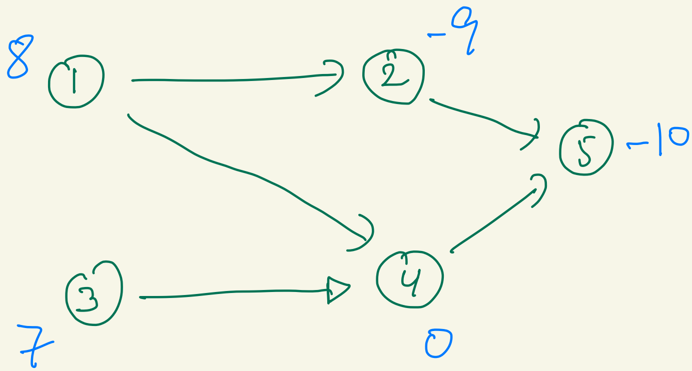
Ahuja 19.2: Maximum weight closures

$D = (V, A, w)$ weighted digraph

$w : V \to \mathbb{R}$

$X \subseteq V$ is a closure if $d^+(X) = 0$



D

Closures in D: (Green = missing in Ahuja)

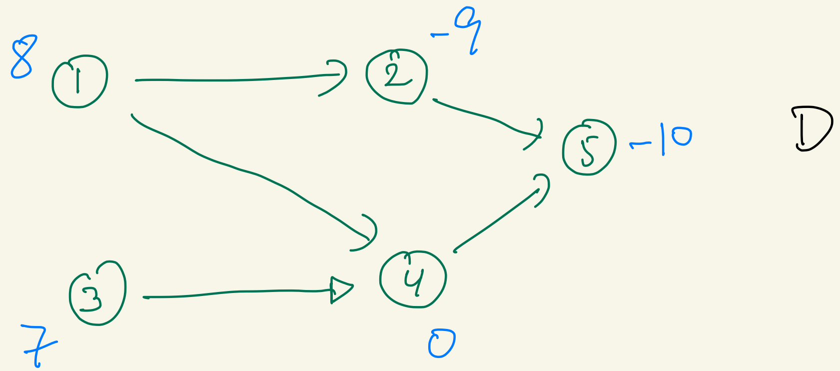
$\emptyset$, $\{1,2,4,5\}$, $\{1,2,3,4,5\}$, $\{2,5\}$, $\{2,4,5\}$, $\{2,3,4,5\}$, $\{4,5\}$, $\{5\}$

Maximum weight closure problem:

Given $D=(V,A,w)$

Find $X \subseteq V$ s.t. $d^+(X)=0$ and

$$w(X)=\sum_{i\in X} w(i) \text{ is maximized}$$

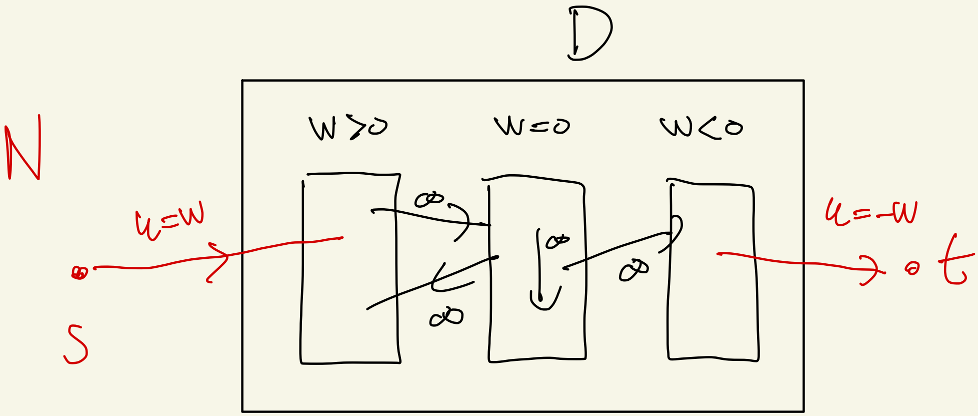


In the example $X=\emptyset$ is the unique maximum closure as

0	-11	-4	-19	-19
$\emptyset$,	$\{1,2,4,5\}$,	$\{1,2,3,4,5\}$,	$\{2,5\}$,	$\{2,4,5\}$,

$\{2,3,4,5\}$,	$\{4,5\}$,	$\{5\}$
-12	-10	-10

Reduction to a max flow problem:



original arcs get ∞ capacity

Claim $u(S,\bar{S})<\infty \iff S-s$ is a closure in D

$(S,\bar{S})$ is a simple (finite) cut if all arcs from S to $\bar{S}$ are incident with s or t.

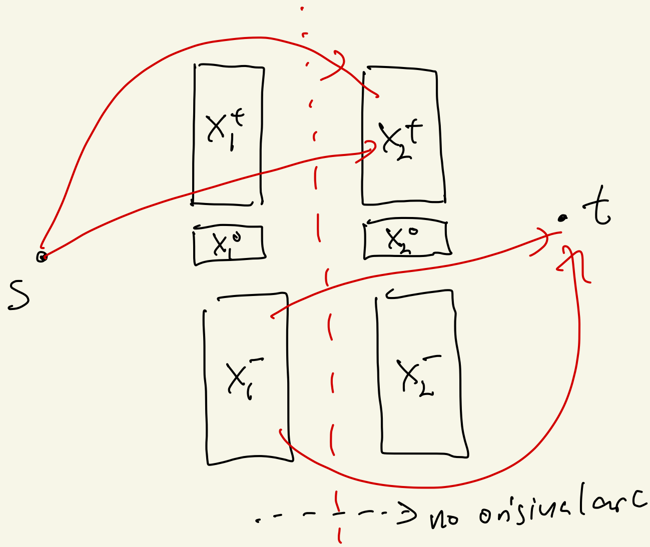
let X_1 be a closure of D and set $X_2 = V \setminus X_1$

$$X_i^+ = \{ j \in X_i \mid w(j) > 0 \}$$

set $$X_i^0 = \{ j \in X_i \mid w(j) = 0 \}$$

$$X_i^- = \{ j \in X_i \mid w(j) < 0 \}$$

$$w(X_1) = \sum_{i \in X_1^+} w(i) - \sum_{i \in X_1^-} |w(i)|$$



$S \in X_1 + s$

$\bar{S} \in X_2 + t$

$$u(S, \bar{S}) = \sum_{j \in X_2^+} w(j) + \sum_{i \in X_1^-} |w(i)|$$

$$\Downarrow$$

$$w(X_1) + u(S, \bar{S}) = \sum_{i \in X_1^+} w(i) - \sum_{i \in X_1^-} |w(i)| + \sum_{j \in X_2^+} w(j) + \sum_{i \in X_1^-} |w(i)|$$

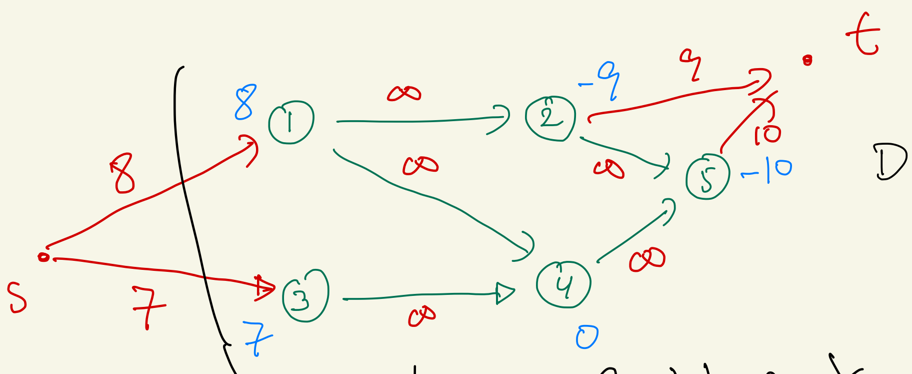
$$= \sum_{\{i \in V \mid w(i) > 0\}} w(i) = \text{constant}$$

$w(X_1) + u(S, \bar{S}) = \text{constant}$

$\Downarrow$ minimizing $u(S, \bar{S})$ is the same as maximizing $w(X_1)$

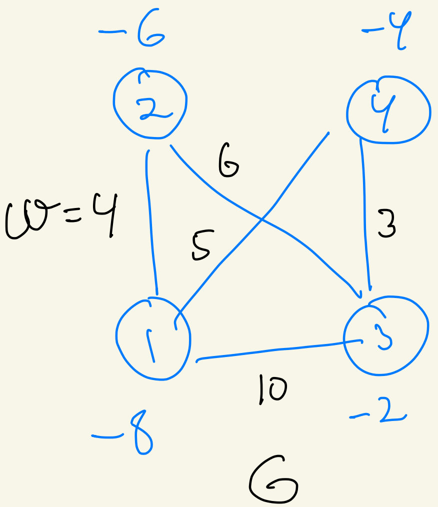
Algorithm:

1. Build N from $D = (V, A, w)$
2. Let x be a max flow in N
3. Let $S = \{v \mid (s, v)\text{-path in } N(x)\}$
4. Return $X = S - s$



unique min cut $\Rightarrow S = \{s\}$ so $X = \emptyset$

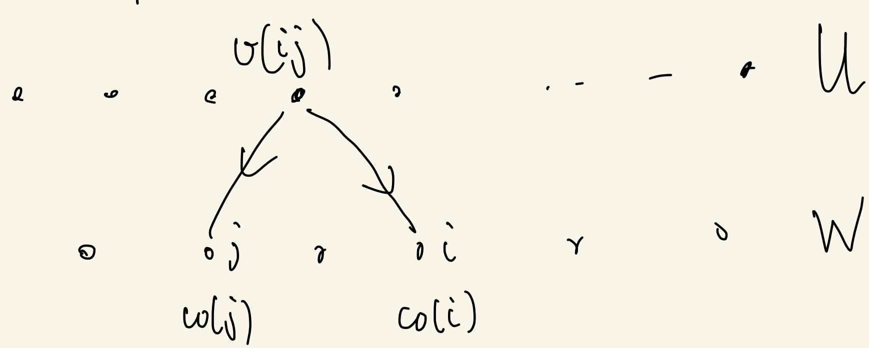
Application 19.2 Selection of freight terminals



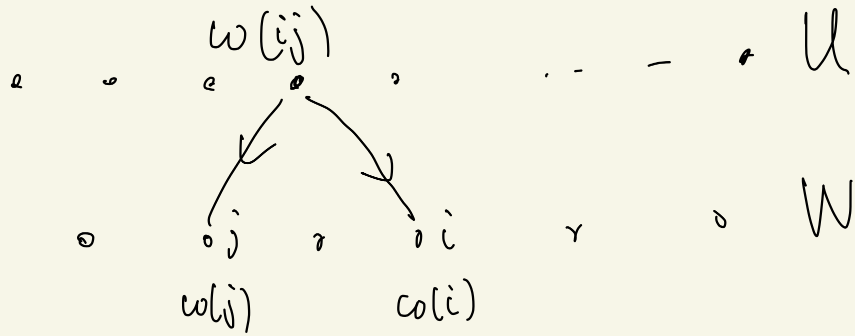
- each node i is a potential terminal with an opening cost $w(i)$
- each edge is a route with a value
- can open route if and only if both end (terminals) are opened

Goal maximize income by opening a subset of the terminals.

create bipartite digraph D_G



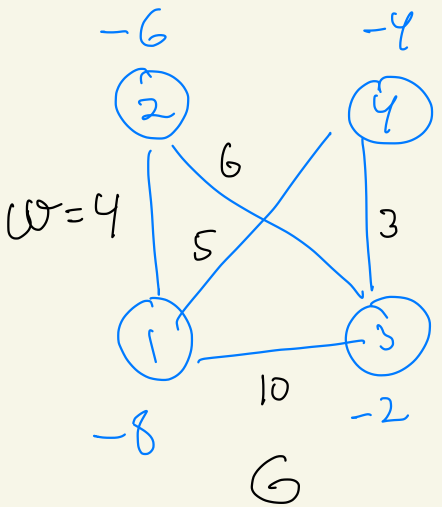
Create bipartite digraph D_G



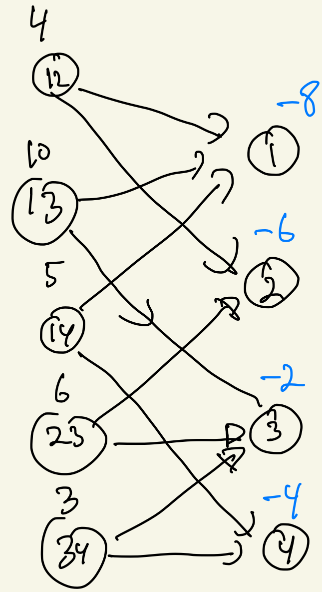
Each closure X in D_G corresponds to a feasible set of terminals to open and

$$w(X) = \sum_{\{ij \in E \mid i,j \in X \cap U\}} w(ij) + \sum_{i \in X \cap W} w(i)$$

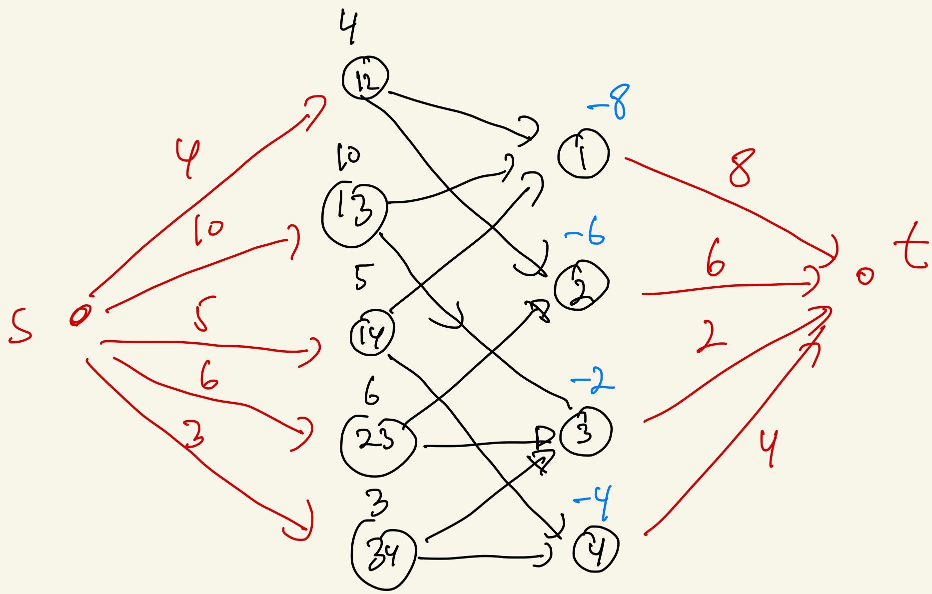
So maximizing the weight of a closure gives a solution in G with max profit



-6
-4
2
4
6
w=4
5
3
1
3
10
-8
-2
G



4
12
-8
10
1
13
-6
5
2
14
-2
6
3
23
-4
3
4
34



4
12
-8
1
4
10
13
8
10
-6
t
5
6
s
5
2
14
2
6
-2
6
3
3
23
4
3
-4
34
4