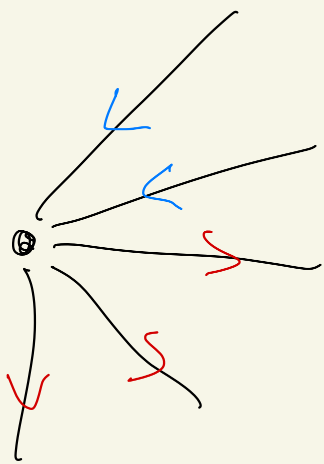
Given $G=(V,E)$ and $V = \{1,2 \dots n\}$

$a_1, a_2 \dots a_n \qquad n=|V|$

<u>Question</u>: can we orient G to a digraph D with $d_D^-(i)=a_i$?

v_1

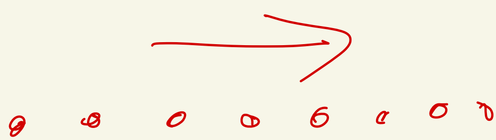


$a_1=2$

<u>step 1</u> orient G arbitrarily

$G \longrightarrow D'$

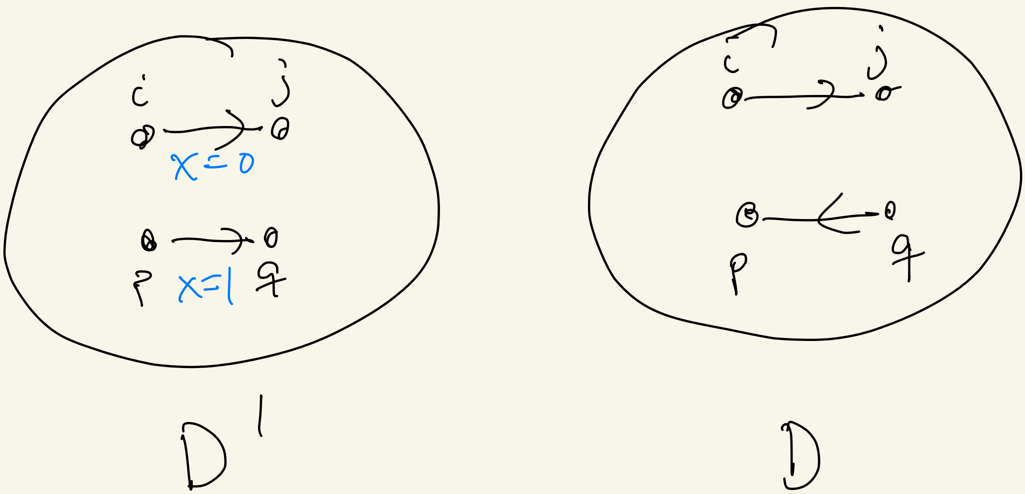
for example



Use D' as a reference orientation

Suppose D is a good orientation

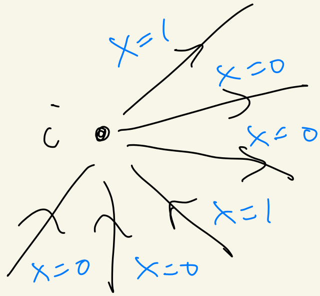
$(d^-_D(v_i) = a_i)$



flow x which points out differences:

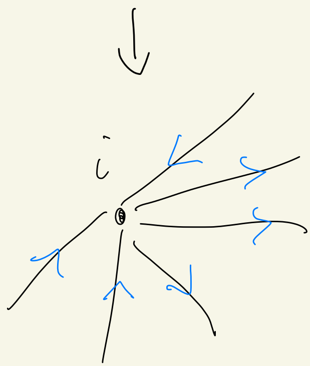
$$x_{ij} = \begin{cases} 1 & \text{if we need to reverse } ij \\ 0 & \text{otherwise} \end{cases}$$

we want $d^-_D(i) = a_i$



in D'

$D(x)$



$$d^-_{D(x)}(i) = d^-_{D'}(i) - \sum_{ji\in A} x_{ji} + \sum_{ij\in A} x_{ij}$$

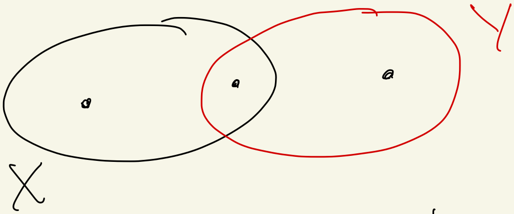
we want

$$a_i = d^-_{D(x)}(i) = d^-_{D'}(i) - \sum x_{ji} + \sum x_{ij}$$

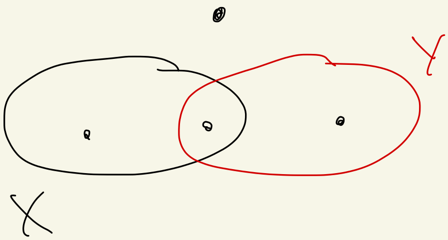
$$\Updownarrow \quad b_x(i) = \sum_{ij\in A} x_{ij} - \sum_{ji\in A} x_{ji} = \underbrace{a_i - d^-_{D'}(i)}_{b(i)}$$

Two sets $X, Y \subseteq V$ are

<u>intersecting</u> if $X \cap Y, X \setminus Y, Y \setminus X \neq \emptyset$



and a pair of intersecting sets X, Y are crossing if also $X \cup Y \neq V$



A family $\mathcal{F}$ of subsets of V is intersecting (crossing) if $X, Y \in \mathcal{F}$ implies $X \cap Y, X \cup Y \in \mathcal{F}$ whenever X and Y are intersecting (crossing)

Let $\mathcal{F}$ be a family of subsets of S and let b be a real valued function defined on $\mathcal{F}$. The function b is **fully submodular** on $\mathcal{F}$ if the inequality

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{1}$$

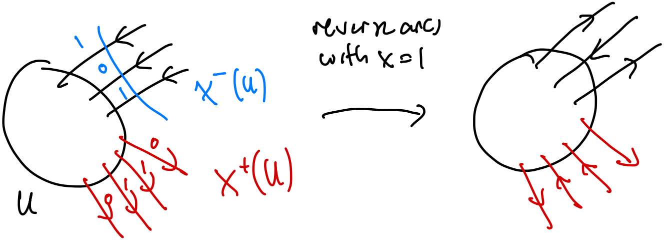
holds for every choice of members X, Y of $\mathcal{F}$. If (1) is only required to hold for intersecting (crossing) members of $\mathcal{F}$, then b is **intersecting (crossing)** submodular on $\mathcal{F}$.

A real valued set function b on S is **modular** if equality holds in (1) for every choice of $X, Y \subseteq S$.

Let $D = (V, A)$ be a directed multigraph let $\mathcal{F}$ be a family of subsets of V such that $\emptyset, V \in \mathcal{F}$ and let b be fully submodular on $\mathcal{F}$. A function $x : A \to \mathcal{R}$ is a **submodular flow** with respect to $\mathcal{F}, b$ if it satisfies

$$x^{-}(U) - x^{+}(U) \leq b(U) \qquad \text{for all } U \in \mathcal{F}. \tag{2}$$

If we take $\mathcal{F} = 2^{\cancel{b}\,V}$ and $b \equiv 0$ we are back at standard circulations (flows).



Theorem 17 (Edmonds and Giles, 1977)

Let $D = (V, A)$ be a directed multigraph. Let $\mathcal{F}$ be a crossing family of subsets of V such that $\emptyset, V \in \mathcal{F}$, let b be crossing submodular on $\mathcal{F}$ with $b(\emptyset) = b(V) = 0$, and let $f \leq g$ be modular functions on A such that $f : A \to \mathcal{Z} \cup \{-\infty\}$ and $g : A \to \mathcal{Z} \cup \{\infty\}$. The linear system

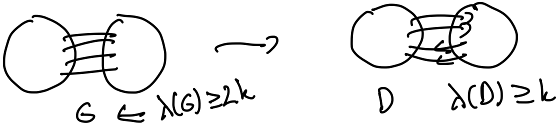
$$\{f \leq x \leq g \text{ and } x^-(U) - x^+(U) \leq b(U) \qquad \text{for all } U \in \mathcal{F}\} \quad (3)$$

is totally dual integral. That is if f, g, b are all integer valued, then the linear program $\min\{c^T x : x \text{ satisfies } (3)\}$ has an integer optimum solution (provided it has a solution). Furthermore, if c is integer valued, then the dual linear program has an integer valued optimum solution (provided it has a solution).

Theorem 18 (Frank 1982, Fujishige 1989)

One can verify in polynomial time whether a given submodular flow problem has a feasible solution. If f, g, b are all integer valued and there exists a feasible submodular flow, then there exist a feasible integer valued submodular flow. Furthermore, if there is also a cost function on the arcs, then one can find a minimum cost feasible submodular flow in polynomial time.

k-arc-strong orientations as a submodular flow problem



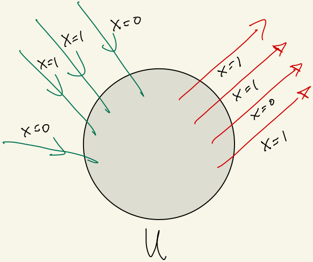
Let $G = (V, E)$ be an undirected graph. Let D be an arbitrary orientation of G. Clearly G has a k-arc-strong orientation if and only if it is possible to reorient some arcs of D so as to get a k-arc-strong directed multigraph.

Suppose we interpret the function $x : A \rightarrow \{0, 1\}$ as follows:

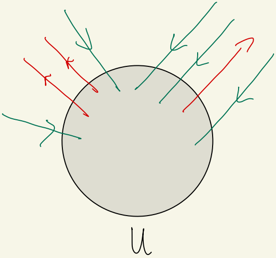
$x(a) = 1$ means that we reorient a in D and

$x(a) = 0$ means that we leave the orientation of a as it is in D.

in D



in D'
after reversals



$$d^-_{D'}(U) = d^-_D(U) + x^+(U) - x^-(U)$$

Then G has a k-arc-strong orientation if and only if we can choose x so that the following holds:

$$d_D^-(U) + x^+(U) - x^-(U) \geq k \qquad \forall\, \emptyset \neq U \subset V. \tag{4}$$

This is equivalent to

$$x^-(U) - x^+(U) \leq (d_D^-(U) - k) = b(U) \qquad \forall\, \emptyset \neq U \subset V \tag{5}$$

$$b(\emptyset) = b(V) = 0. \tag{6}$$

$0 \leq x(a) \leq 1 \quad \forall a \in A$

Observe that the function b is crossing submodular on $\mathcal{F} = 2^V$ (it is not fully submodular in general, since we have taken $b(\emptyset) = b(V) = 0$).

Thus we have shown that G has a k-arc-strong orientation if and only if there exists a feasible integer valued submodular flow in D with respect to the functions $f \equiv 0, g \equiv 1$ and b.

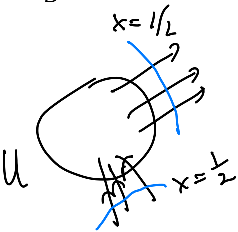
Recall Nash-Williams orienta~~t~~tion theorem

Theorem 19 (Nash-Williams, 1960)

A multigraph G has a k-arc-strong orientation if and only if G is $2k$-edge-connected.

Proof: (Frank 1984, Jackson 1988) Suppose that G is $2k$-edge-connected, that is $d_G(X) \geq 2k$ for all proper non-empty subsets of V (by Menger's theorem). Let D be arbitrary orientation of G. We claim that $x \equiv \frac{1}{2}$ is a feasible submodular flow. This follows from the following calculation:

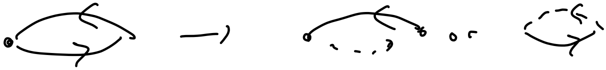
$$\begin{aligned} d_D^-(U) + x^+(U) - x^-(U) &= d_D^-(U) + \frac{1}{2}d_D^+(U) - \frac{1}{2}d_D^-(U) \\ &= \frac{1}{2}d_D^-(U) + \frac{1}{2}d_D^+(U) \\ &\geq \frac{1}{2}(2k - d_D^+(U)) + \frac{1}{2}d_D^+(U) \\ &= k. \end{aligned} \tag{7}$$



Hence it follows from the integrality statement of Theorem 18 and the equivalence between (4) and (5) that there is a feasible integer valued submodular flow x in D with respect to f, g and b. As described above this implies that G has a k-arc-strong orientation where the values of x prescribe which arcs to reverse in order to obtain such an orientation from D. □

Theorem 20 (Jackson 1988)

Every 2k-arc-strong digraph contains a spanning k-arc-strong oriented graph.



proof Let $D = (V, A \cup E)$

where A is the set of arcs that are not in 2-cycles and

E is the arc set of the 2-cycles in D

- By assumption $\lambda(D) \geq 2k$

  so $d^-_D(u) + d^-_E(u) \geq 2k$

- Denote by D_o the oriented subdigraph spanned by the arcs in A

  so $D_o = (V, A)$

- Let $D' = (V, A')$ where A' is obtained from E by deleting one arc of every 2-cycle (arbitrarily)

- Every k-arc-strong oriented spanning subdigraph of D can be obtained by reversing 0 or more arcs in A' (those in A are fixed)

- Interpret a flow x on A' by

$$x(a) = \begin{cases} 1 & \to \text{ reverse } a \\ 0 & \to \text{ keep } a \end{cases}$$

- We want

$$(*) \quad d^-_D(U) + d^-_{D'}(U) + x^+(U) - x^-(U) \geq k$$

for all $\emptyset \neq U \subset V$

$$d^-_D(U) + d^-_{D'}(U) + x^+(U) - x^-(U) \geq k$$

$\Updownarrow$

$$x^-(U) - x^+(U) \leq \left(d^-_D(U) + d^-_{D'}(U)\right) - k$$

$$= \hat{b}(U)$$

submodular

$\boxed{W}$

Extend x from A' to $A \cup A'$
by $f(a) = g(a) = 0 \quad \forall a \in A$
then D has a k-arc-strong
spanning oriented subdigraph
if and only if there exists a
feasible 0,1 solution to

$$x^-(U) - x^+(U) \leq \hat{b}(U)$$

$$f(a) \leq x(a) \leq g(a) \quad \forall a \in A \cup A'$$

$$x^-(U) - x^+(U) \le \hat{b}(U)$$

$$f(a) \le x(a) \le g(a) \quad \forall a \in A \cup A'$$

Claim

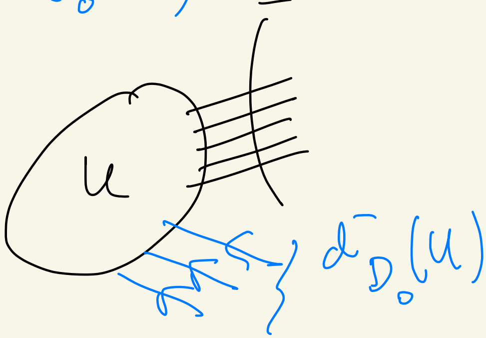
$$x(a) = \begin{cases} 0 & \text{if } a \in A \\ 1/2 & \text{if } a \in A' \end{cases}$$

is a feasible solution

$$d^-_{D_0}(U) + d^-_{D'}(U) + x^+(U) - x^-(U)$$
$$= d^-_{D_0}(U) + d^-_{D'}(U) + \frac{1}{2} d^+_{D'}(U) - \frac{1}{2} d^-_{D'}(U)$$
$$= d^-_{D_0}(U) + \frac{1}{2}\left(d^+_{D'}(U) + d^-_{D'}(U)\right)$$
$$\ge d^-_{D_0}(U) + \frac{1}{2}\left(2k - d^-_{D_0}(U)\right)$$

$$\ge 2k - d^-_{D_0}(U)$$

$$\ge k + \frac{1}{2} d^-_{D_0}(U)$$
$$\ge k$$



We have shown that

$x \equiv \frac{1}{2}$ on arcs in A'

and $x \equiv 0$ on arcs in A

is a feasible submodular flow

on $\hat{D} = (V, A \cup A')$ with bounds

$f(a) = 0 \quad \forall a \in A \cup A'$

$g(a) = 0 \quad \forall a \in A$

$g(a') = 1 \quad \forall a' \in A'$

so by the integrality thm

there exists a 0,1 solution x'

and this gives the desired

orientation: reverse $a \in A'$

precisely if $x'(a) = 1$

Notice that by formulating the problem above as a minimum cost submodular flow problem, we can also solve the weighted version where the two possible orientations of an edge may have different costs and the goal is to find the cheapest k-arc-strong orientation of the graph. This clearly includes the problem where we wish to find the minimum number of arcs to reverse in order to obtain a k-arc-strong directed multigraph, hence we have

Theorem 21 (Frank 1982)

Given a directed multigraph D, one can find in polynomial time the minimum number of arcs whose reversal in D results in a k-arc-strong directed multigraph.

This includes the case when D has no such reversal which can be detected by checking whether the submodular flow problem above has a feasible solution.