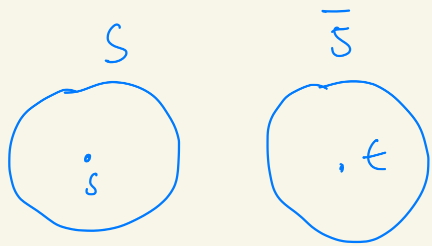
Maximum flow problem

Given a network $N = (V \cup \{s,t\}, A, \ell \equiv 0, u)$

Find an (s,t)-flow x s.t. $b_x(s)$ is maximum

<u>(s,t)-cut :</u>



<u>Lemma 3.5.1</u> For every (s,t)-flow in N and for every (s,t)-cut $(S,\bar{S})$

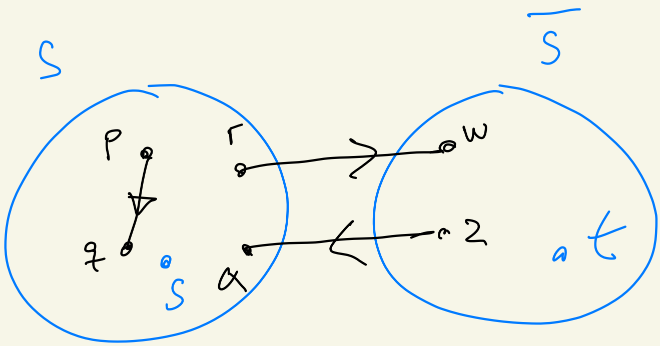
$$b_x(s) = x(S,\bar{S}) - x(\bar{S},S)$$

$$b_x(S) = b_x(s) + \sum_{i \in S - s} b_x(i)$$

$$= \sum_{i \in S} b_x(i)$$

$$= \sum_{i \in S} \left(\sum_{ij \in A} x_{ij} - \sum_{ji \in A} x_{ji} \right)$$

$$= x(S, \bar{S}) - x(\bar{S}, S)$$



$$\underline{b_x(s)} = x(S, \bar{S}) - x(\bar{S}, S)$$

$$\leq u(S, \bar{S}) - 0 = \underline{u(S, \bar{S})}$$

Note

if $\delta_x(S) = u(S, \bar{S})$

then x is a maximum flow

and $(S, \bar{S})$ is a minimum

(s,t)-cut i.e. $u(S,\bar{S})$ is

minimum over all (s,t)-cuts

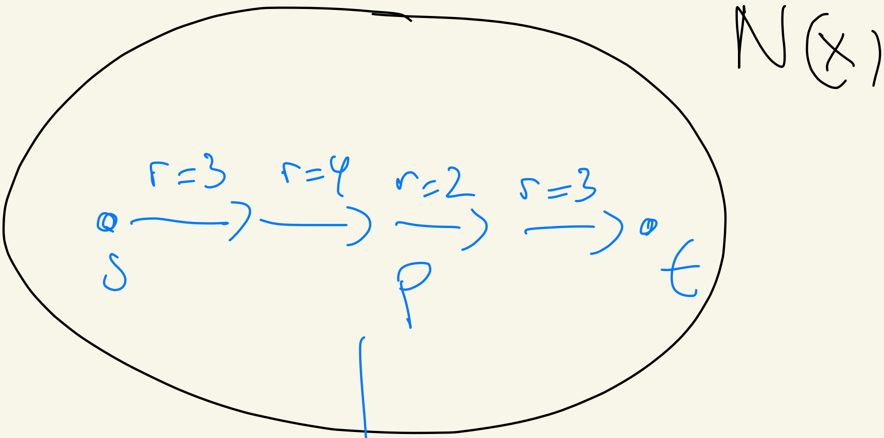
Suppose x is an (s,t)-flow in $N = (V \cup \{s,t\}, A, l \equiv 0, u)$

and P is an (s,t)-path in $N(x)$

s.t. $\min\{r_{ij} \mid ij \text{ is an arc of } P\} = k$

then let $f(P)$ be the path flow that sends k units along P in $N(x)$

then $x' = x \oplus f(P)$ is an (s,t)-flow of value $b_x(s) + k > b_x(s)$



$k = 2$

$f(P)$ has value 2

$\Downarrow$

$$b_{x'}(s) = b_x(s) + 2$$

$$(x' = x \oplus f(P))$$

$$\delta(P) = \min\{r_{ij} \mid ij \in A(P)\}$$

directed path in $N(x)$

$\delta = 2$

$$s \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ\, t$$

corresponds to

$$s \circ \xrightarrow{x<u} \circ \xrightarrow{x<u} \circ \xleftarrow{x>0} \circ \xrightarrow{x<0} \circ\, t$$

forward   backward

Thm (Max Flow - Min Cut)

Let $N = (V \cup \{s,t\}, A, \ell \equiv 0, u)$ be an (s,t)-network and let x be a feasible flow. Then

(a) x is a maximum flow

$\Updownarrow$

(b) $\nexists$ (s,t)-path in $N(x)$

$\Updownarrow$

(c) $\exists$ an (s,t)-cut $(S,\bar{S})$ s.t. $u(S,\bar{S}) = b_x(s)$

P: (c) $\Rightarrow$ (a) ✓

(a) $\Rightarrow$ (b)

$\neg$(b) $\Rightarrow$ $\neg$(a)

Suppose there is an (s,t)-path P in $N(x)$ and let $f(P)$ be a path flow of value $\delta(P)$

then $x' = x \oplus f(P)$ is also an (s,t)-flow and

$$b_{x'}(s) = b_x(s) + \delta(P)$$

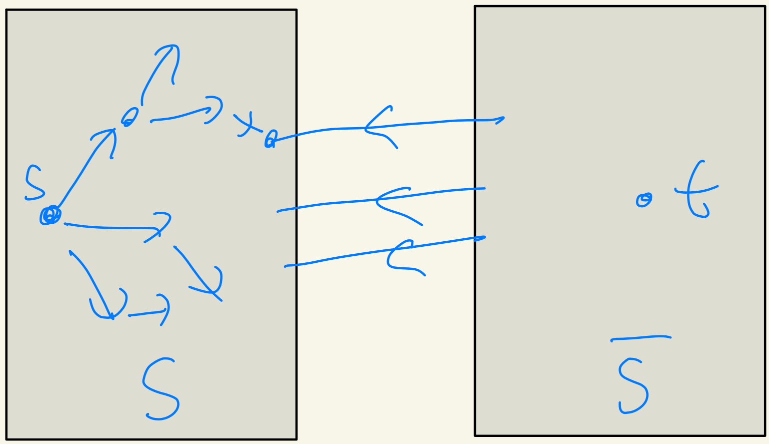
$\Downarrow$

x is not maxi

(b) => (c)

Suppose (b) holds that is
t cannot be reached from s
in $N(x)$

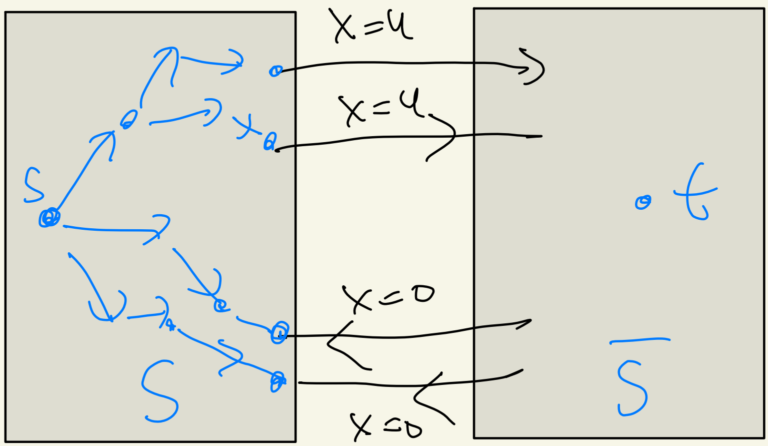
$S = \{v \mid \exists (s,v)\text{-path in } N(x)\}$



only $\bar{S} \to S$ arcs in $N(x)$

in N

blue arcs are in N(x)



$$b_x(S) = x(S,\bar{S}) - x(\bar{S},S)$$

$$= u(S,\bar{S}) - 0$$

$$= u(S,\bar{S})$$

so (c) holds

Ford Fulkerson method

1. $x \equiv 0$

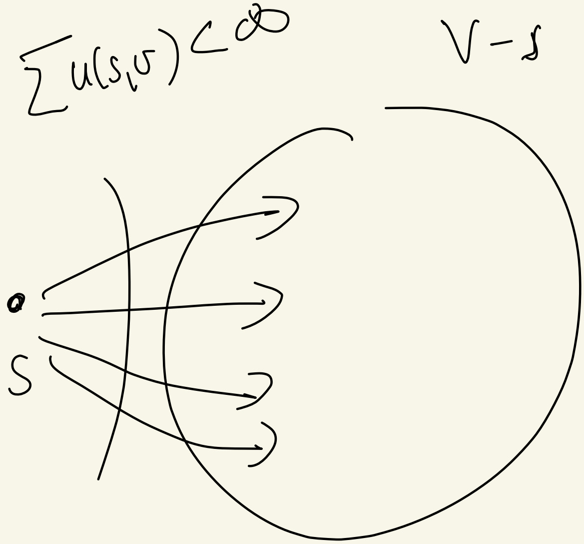
2. while $\exists$ (s,t)-path in $N(x)$

   let P be such a path

   $x \leftarrow x \oplus f(P)$

   path flow of value $\delta(P)$

if all capacities are integer then the method terminates with a maximum flow



Integrality theorem

if $N = (V \cup \{s,t\}, A, l \equiv 0, u)$ satisfies that $u_{ij} \in \mathbb{Z}_0$ $\forall ij \in A$ then there exists an integer-valued maximum (s,t)-flow.