Thm I

Integrality theorem for (maximum) $(s,t)$ flows:

Let $N = (V \cup \{s,t\}, A, l \equiv 0, u)$ satisfy that $u_{ij} \in \mathbb{Z}_+ \ \forall ij \in A$.

Then $N$ has integervalued flows

$x_0, x_1 \ldots x_k$ when

$k$ is the value of a maximum flow in $N$ and $|x_i| = i$ $(|x| = b_x(s))$

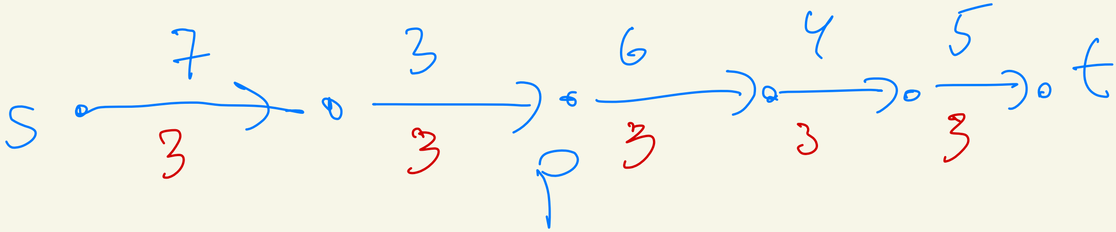
$x_0$ exists $x \equiv 0$ is a good choice

$N(x_0) = N$

if $k > 0$ then $\exists (s,t)$-path $P$ in $N(x_0)$

$\delta(P) =$ min capacity along $P$:

$r_{ij}$



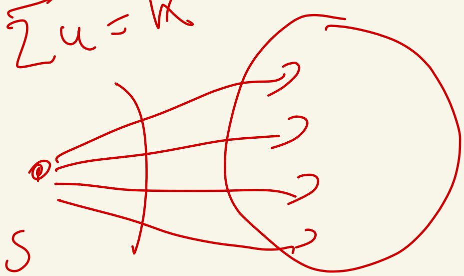

$\delta(P) = 3$

current $x$ $\qquad$ $x'$ flow in $N(x)$

then $x'' = x \oplus x'$ is a feasible flow in $N$ with

$$\delta_{x''}(v) = \delta_x(v) + \delta_{x'}(v)$$

$\sum u = k$

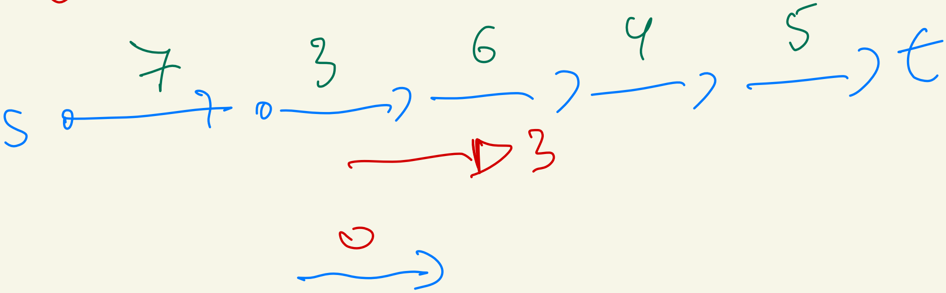


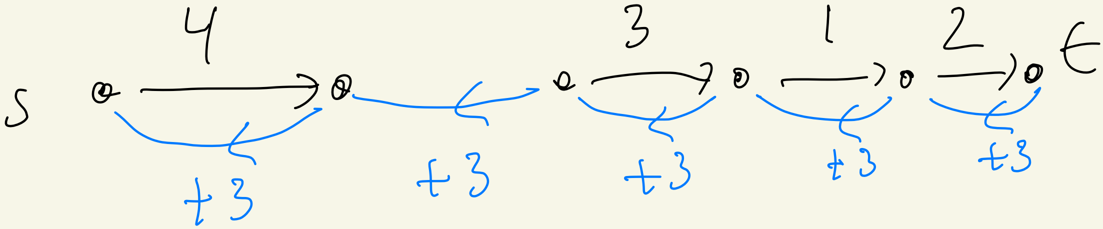

maxflow $\le k$

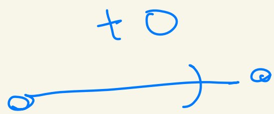
Note that if $u_{ij} \in \mathbb{Z}_+$ $\forall ij \in A$ and $\tilde{x}$ is an integer flow in $N(x)$ and $x$ is integer flow in $N$ then $\hat{x} = x \oplus \tilde{x}$ is an integer flow in $N$ and all capacities in $N(\hat{x})$ are integers.

Recall $r_{ij} = (u_{ij} - x_{ij}) + x_{ji}$

Going from $N(x)$ to $N(\hat{x})$ (when $\tilde{x}$ is a path flow from s to t)






Back to proof of Theorem I

suppose we have found $x_i$ $i<k$

Let $P$ be an $(s,t)$-path in $N(x_i)$

obtain $x_{i+1}$ by sending one unit along $P$ in $N(x_i)$

Note that $N(x')$ has all capacities integral (all in $\mathbb{Z}_+$).

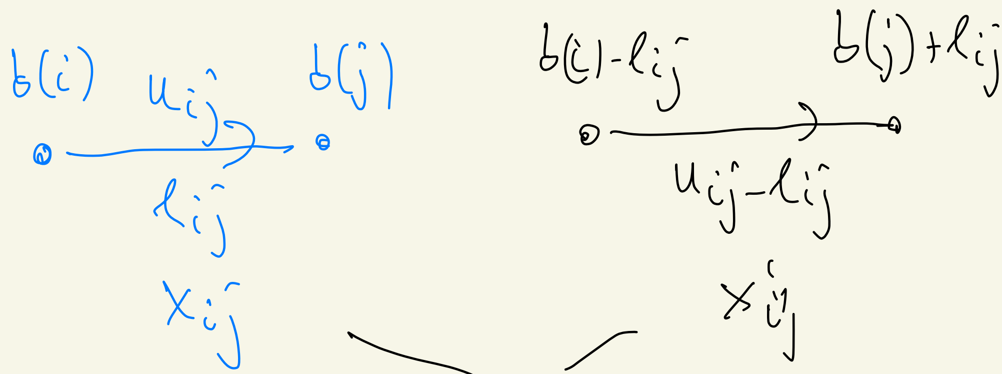
$\in \mathbb{Z}.$

$$r_{ij} = (u_{ij} - x_{ij}) + x_{ji}$$

## Corollary if

$$N = (V, A, \ell, u, b)$$

satisfies that $\ell$, $u$ and $b$ are all integer valued and there is feasible flow in N then $\exists$ a feasible integer valued flow in N.




$$x_{ij} = x'_{ij} + \ell_{ij}$$

$\delta > 0$ $\delta = 0$ $\delta < 0$