Extension of the '⊕' operator to allow lowerbounds

$$X \oplus \tilde{X} \qquad \tilde{X} \in N(X)$$ netto flow

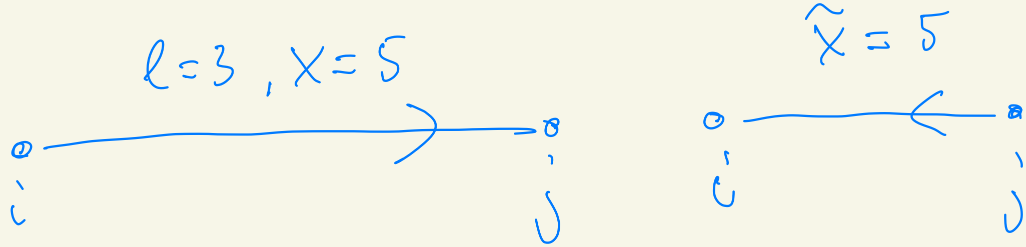
(a) $i \xrightarrow{x_{ij} \geq 0} j \qquad i \xrightarrow{\tilde{x}_{ij} > 0} j$

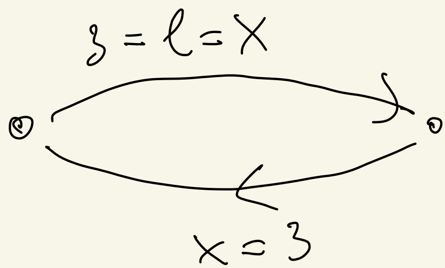
$x_{ji} = 0$

$$\downarrow$$

$$i \xrightarrow{x_{ij} + \tilde{x}_{ij}} j$$

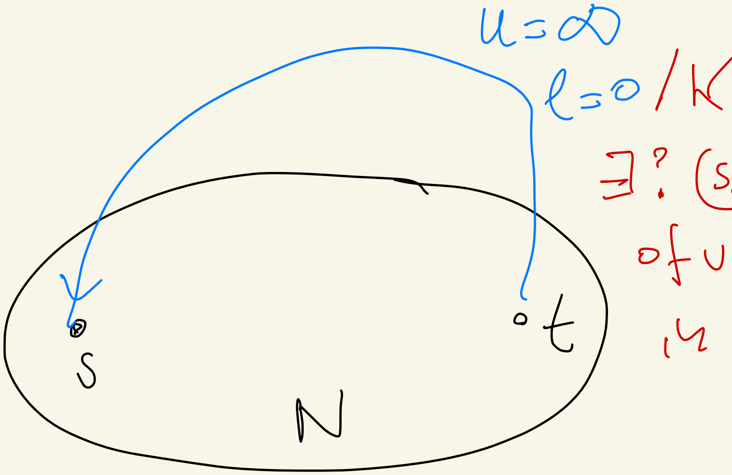
(b)+(c) $i \xrightarrow{x_{ij} \geq 0} j \qquad i \xleftarrow{\tilde{x}_{ji} > 0} j$





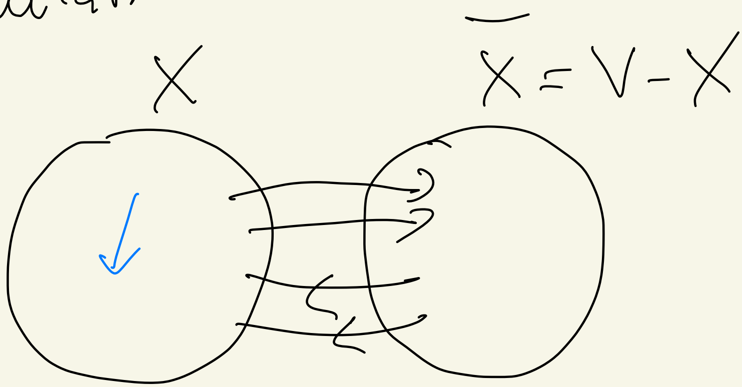
$$r_{ji} = (u_{ji} - x_{ji}) + (x_{ij} - \ell_{ij})$$

(s,t)-flows → circulation



∃? (s,t)-flow of value k in N

Bad situation for existence of circulation:



$$\left.\begin{aligned} 0 &= x(X,\bar{X}) - x(\bar{X},X) \\ &\le u(X,\bar{X}) - l(\bar{X},X) \end{aligned}\right\} \quad l(\bar{X},X) \le u(X,\bar{X})$$

Theorem Hoffman

Let $N = (V, A, \ell, u)$

Then N has a feasible circulation if and only if

$$\ell(\bar{S}, S) \leq u(S, \bar{S}) \quad \forall S \subseteq V$$

$$\emptyset, V = S$$

P: $\Rightarrow$ if x is feasible circulation then $\ell(\bar{S}, S) \leq u(S, \bar{S})$

(saw this on previous slide)

$\Leftarrow$ suppose $\ell(\bar{S}, S) \leq u(S, \bar{S})$ for all $\emptyset, V \neq S \subseteq V$

Idea: construct a feasible circulation step by step.

Start with $x \equiv 0$ circulation ✓

If x is feasible we are done

so assume $\exists\, ij \in A$ s.t

$$x_{ij} < l_{ij}$$

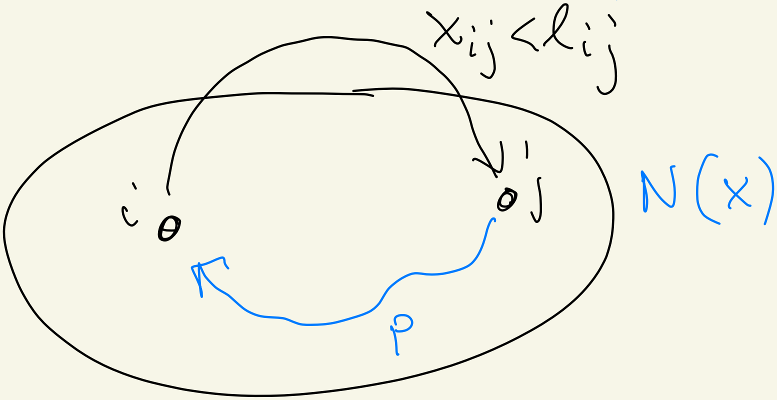
Try to find an (j,i)-path P in $N(x)$ and add a suitable path flow $f(P)$ to x:

$$x \Leftarrow x \oplus \delta(P)$$

Case 1 $0 < \delta(P) < l_{ij} - x_{ij}$

add path flow of value $\delta(P)$ along P to x:

$$x \leftarrow x \oplus \delta(P)$$



Case 2 $\delta(P) \geq l_{ij} - x_{ij}$

let $\delta'(P) = l_{ij} - x_{ij}$

$$x \leftarrow x \oplus \delta'(P)$$

In Case 2 the arc ij is now ok (new x satisfies

$x_{ij} \geq l_{ij}$)

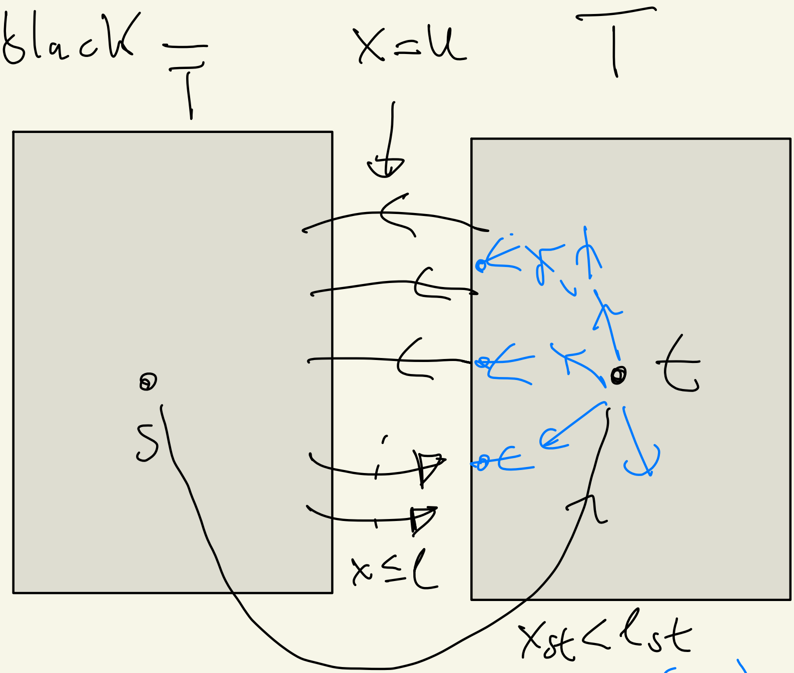
In Case 1 we repeat and try to find a new (j,i)-path in $N(x)$ where x is the new flow

Suppose none of Case 1 and 2 occur for some arc st where $x_{st} < l_{st}$

so no (t,s)-path in $N(x)$

N(x) blue

N black



$T = \{ v \mid \exists\, (t,v)\text{-path in } N(x) \}$

$s \notin T$ (as not in Case 1 or 2)

Now:

$$\underline{u(T,\overline{T})} = x(T,\overline{T}) = x(\overline{T},T) < \underline{\ell(\overline{T},T)}$$

Contradiction ↯