Minimum value (s,t)-flows

Given $N = (V \cup \{s,t\}, A, \ell, u)$

Find (s,t)-flow x which is feasible ($\ell_{ij} \leq x_{ij} \leq u_{ij} \;\; \forall ij \in A$)

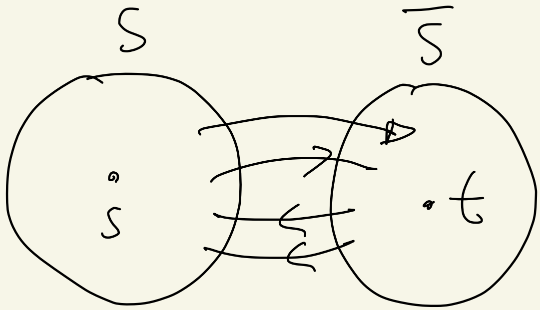
and which minimizes

$|x| = b_x(s)$

Note: there may be no feasible (s,t)-flow

$$s \xrightarrow{\ell=3} \cdot \xrightarrow{u=2} t$$

(s,t)-cut



$$|x| = \delta_x(S) = x(S,\bar{S}) - x(\bar{S},S)$$
$$\geq \ell(S,\bar{S}) - u(\bar{S},S)$$
$$= \gamma(S,\bar{S})$$ demand of the cut $(S,\bar{S})$

$$|x| \geq \max\{\gamma(S,\bar{S}) \mid (S,\bar{S})\ (s,t)\text{-cut}\}$$

<u>Thm 3.9.1</u>

$$\min_{x \text{ feasible}} |x| = \max\{\gamma(S,\bar{S}) \mid s \in S, t \in \bar{S}\}$$

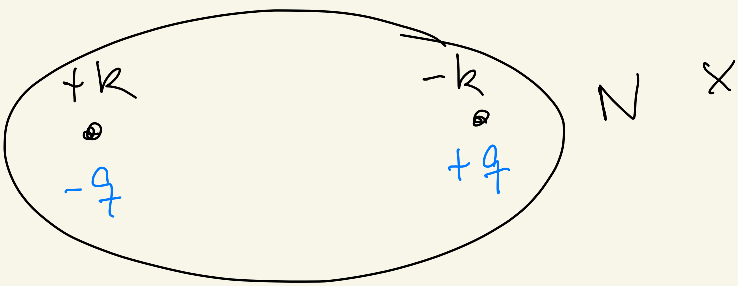
observe:

if y is any (t,s)-flow in $N(x)$

then $x' = x \oplus y$ is a feasible (s,t)-flow in N of value

$$|x'| = \underset{k}{|x|} - \underset{q}{|y|}$$

(recall $b_{x'} \equiv b_x + b_y$



$k-q$ $\qquad$ $-k+q$

suppose $\check{X}$ is a minimum value (s,t)-flow then

$$\check{X} = X \oplus Z \quad \text{when } Z \text{ is a flow in } N(X)$$

Z is a (t,s)-flow

$(\delta_{\check{X}} \equiv \delta_X + \delta_Z)$

This implies that we can find a minimum value (s,t)-flow in N by

1. Finding a feasible (s,t)-flow x
2. Finding a maximum value (t,s)-flow in $N(x)$

Suppose now that y is a maximum value (t,s)-flow in $N(x)$

MaxFlowMinCut thm $\rightarrow$ (t,s)-cut $r(T,\bar{T})$ is minimum

$$|y| = r(T,\bar{T})$$

$$= \sum_{\substack{i \in T \\ j \in \bar{T}}} r_{ij}$$

$$= \sum_{\substack{i \in T \\ j \in \bar{T}}} (u_{ij} - x_{ij}) + (x_{ji} - l_{ji})$$

$$= \sum_{\substack{i \in T \\ j \in \bar{T}}} (u_{ij} - l_{ji}) + \sum_{\substack{i \in T \\ j \in \bar{T}}} (x_{ji} - x_{ij})$$

$$|y| = \sum_{\substack{i \in T \\ j \in \bar{T}}} (u_{ij} - \ell_{ji}) + \sum_{\substack{i \in T \\ j \in \bar{T}}} (x_{ji} - x_{ij})$$

$$= \left[u(T, \bar{T}) - \ell(\bar{T}, T) \right]$$

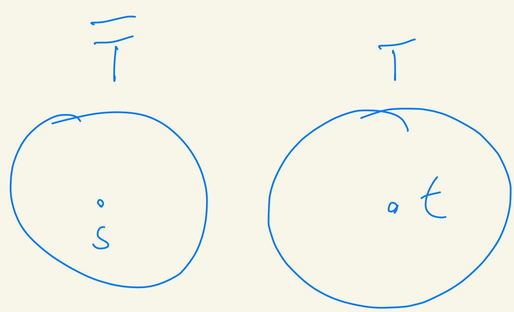
$$+ \left[x(\bar{T}, T) - x(T, \bar{T}) \right]$$

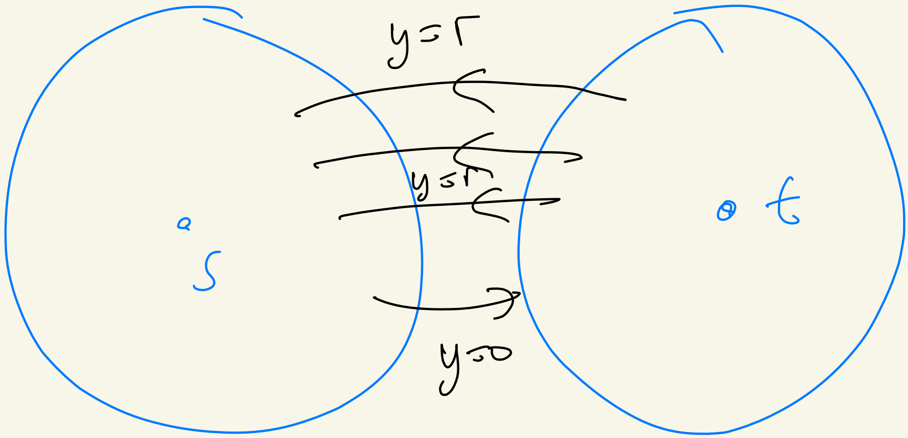
$$= -\gamma(\bar{T}, T) + |x|$$

$$\Downarrow$$

$$\gamma(\bar{T}, T) = |x| - |y| = |x'|$$

$$x' = x \oplus y$$




y = r
y = r
y = 0
s
t

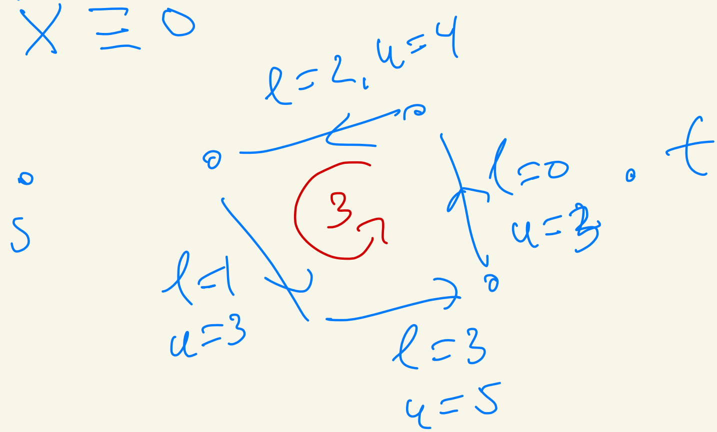
Conclusion

x feasible (s,t)-flow

if $N(x)$ has a (t,s)-flow of value at least $|x|$, then we can obtain a feasible (s,t)-flow $\tilde{x}$ of value 0 (a circulation)

NB: Not the same as saying

$$\tilde{x} \equiv 0$$



Path-covering problem for acyclic digraphs

Given D acyclic find

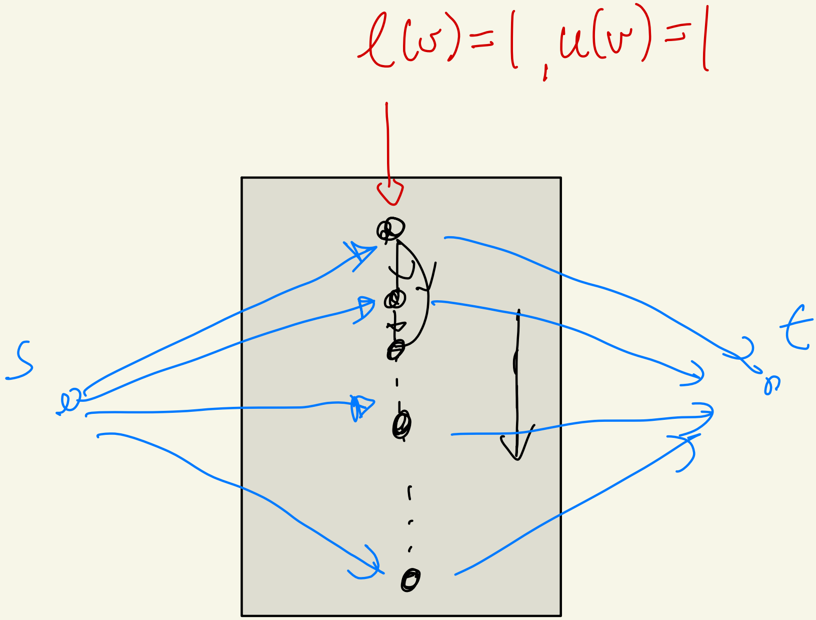
$P_1, P_2, \ldots, P_k$ $\quad k \geq 1$

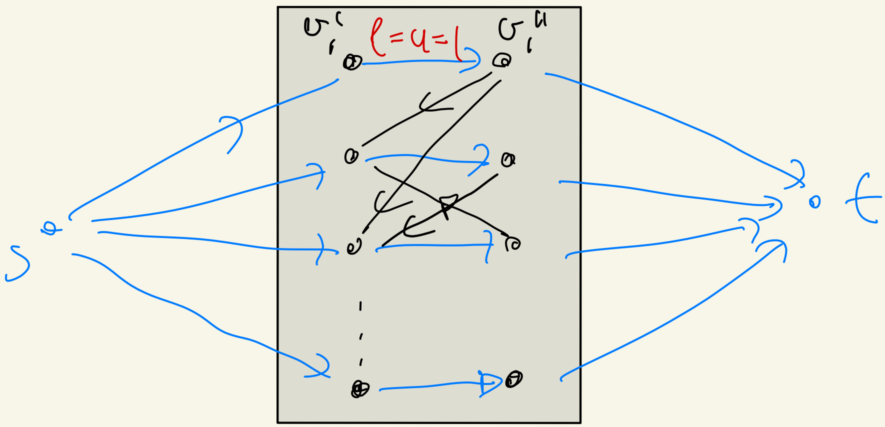
s.t. $V(P_i) \cap V(P_j) = \emptyset \quad i \neq j$

and $V(D) = \bigcup_{i=1}^{k} V(P_i)$

and k is minimum.

Note $k = 1 \iff D$ has a hamilton path





N_D

<u>claim</u> N_D has an (s,t)-flow of value k

$\Updownarrow$ $\exists\, P_1 \dots P_k$ covering $V(D)$

P:

Given $P_1, P_2 \ldots P_k$ we send 1 unit along each of the P_i' corresponding (s,t)-paths in N_D

$\left(P_1 = v_1 v_2 v_3 \, , \, P' = s v_1' v_1'' v_2' v_2'' v_3' v_3'' t \right)$

the result is a feasible (s,t)-flow of value k

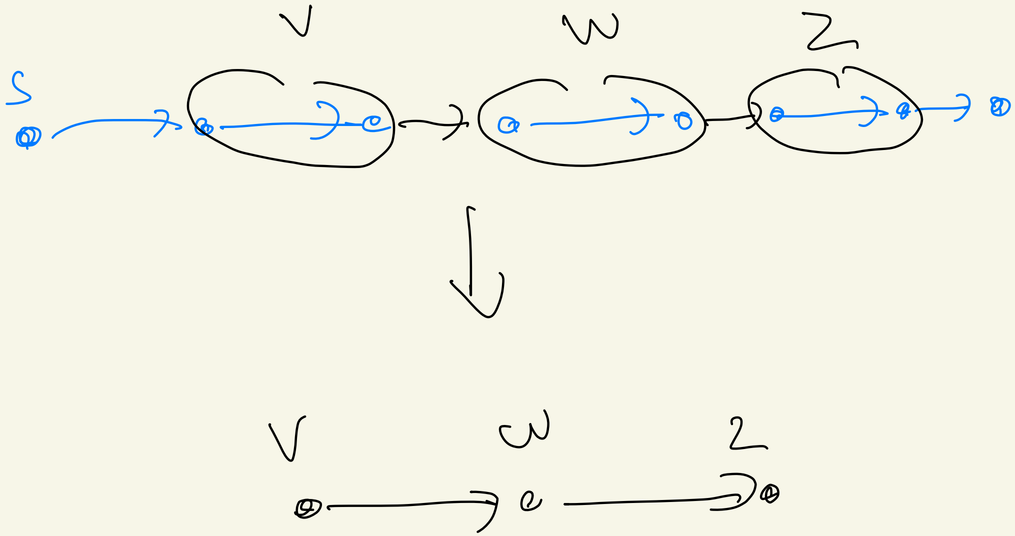
Conversely: Let x be feasible integer flow in N_D

$|x| = q \in \mathbb{Z}$

x decomposes into q path flows each of value 1 (by flow decomp thm)

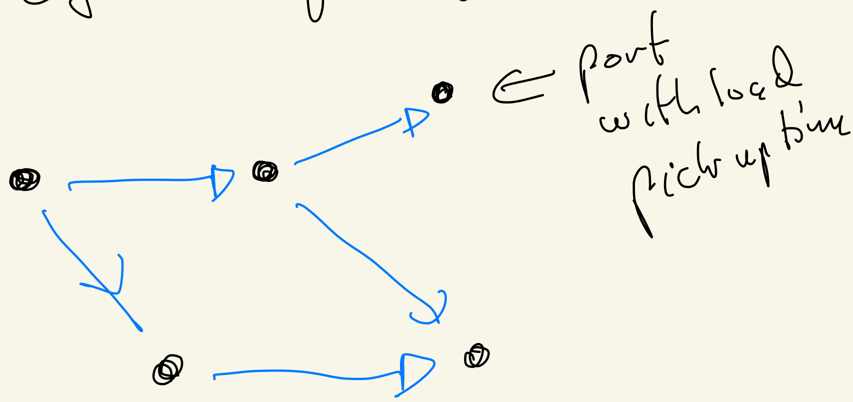
Let $P_1', P_2' \ldots P_q'$ be paths along which we decompose. Then $V(P_i') \cap V(P_j') = \emptyset$

$\Rightarrow P_1, P_2 \ldots P_q$ vertex disj in D and cover $V(D)$

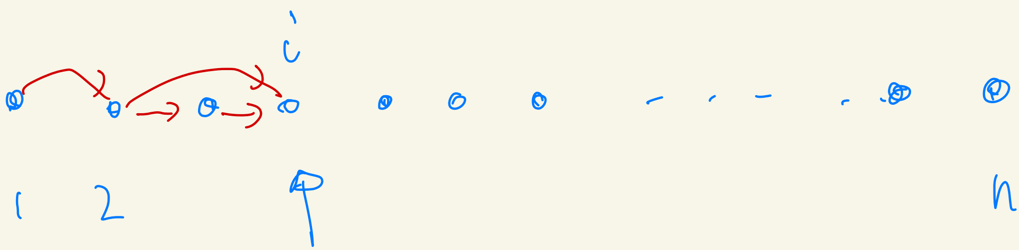


v
w
z
s
v
w
z

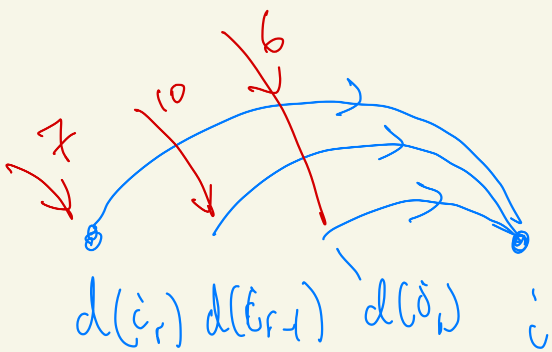
Application 6.6 in section 6.2 of Ahuja. (tanker scheduling)



longest paths in acyclic digraph



$d(i)$ = length of longest path ending in i



$$d(i) = \max\{d(i_j) \mid i_j \to i \in A\} + 1$$

longest path has length

$$\max\{d(i) \mid i \in [n]\}$$