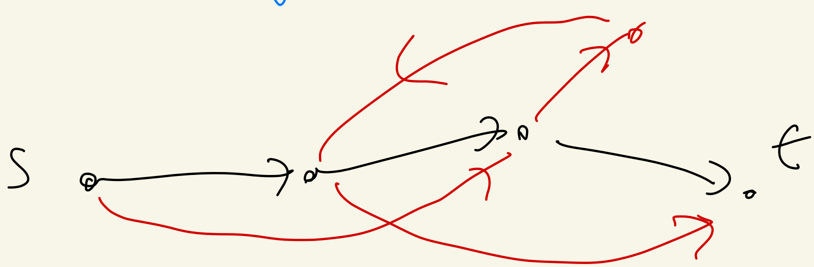
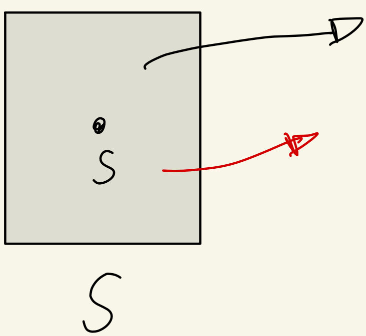
Combinatorial application of the max flow min cut theorem

The arc-connectivity from s to t in a digraph $D=(V,A)$ with $s,t \in V$, called $\lambda(s,t)$ is the maximum number of arc-disjoint (s,t)-paths in D





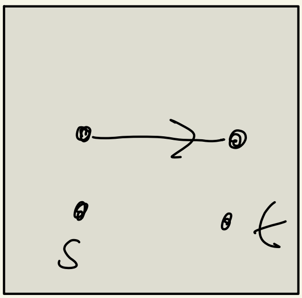
$$d^+(S) \geq \lambda(s,t)$$

Theorem (Menger)

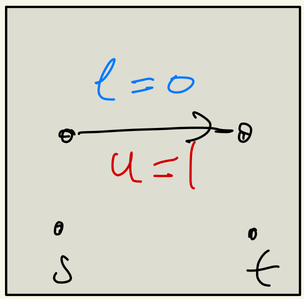
Let $D=(V,A)$ be a digraph and let $s,t \in V$. Then

$$\lambda(s,t) = \min\{d^+(S) \mid s \in S, t \in V-S\}$$

P:



$D=(V,A)$



$N(D)$

$V = V' \cup \{s,t\}$

$$N(D) = \{V' \cup \{s,t\}, A, \ell, u\}$$

Claim: The maximum value of an (s,t)-flow in $N(D)$ is $\lambda(s,t)$

$\underline{\max |x| \geq \lambda(s,t):}$

Given $P_1, P_2 \dots P_k$ arc-disj (s,t)-paths in D

Send 1 unit of flow along each P_i in $N(D)$ $\rightarrow$ gives a feasible (s,t)-flow x with $|x| = k$

$\underline{\lambda(s,t) \geq |x^*|:}$ x^* max flow

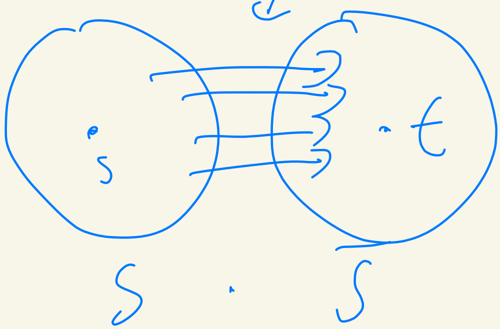
Given an integer valued maximum flow x^*

Decompose x^* into path flows and cycle flows $f(P_1') \dots f(P_{|x^*|}), f(C_1) \dots f(C$

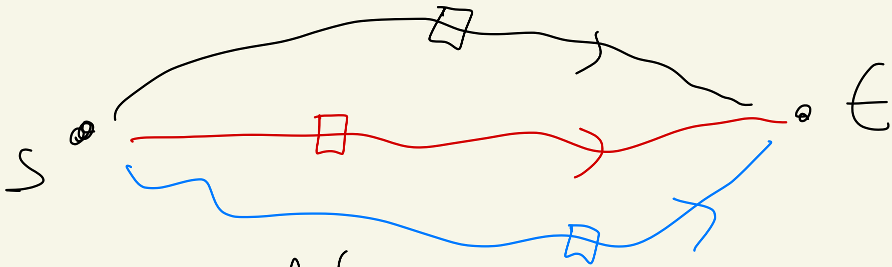
each P_i' is an (s,t)-flow

Each P_i' is a path (the same in D and they are arc-disjoint (since $u_{ij} = 1 \ \forall ij \in A$)

$$\underline{\lambda(s,t)} = |x^*| = u(S,\bar{S}) = \underline{d^+(S)}$$



Internally disjoint (s,t)-paths in digraphs



$$\kappa(s,t) \stackrel{\text{def}}{=} \text{max \# of internally disjoint } (s,t)\text{-paths}$$

assume $s \not\to t$

(s,t)-separator is a set $X \subseteq V - \{s,t\}$ s.t. $D - X$ has no (s,t)-path (that is, every (s,t)-path uses at least one vertex in X)

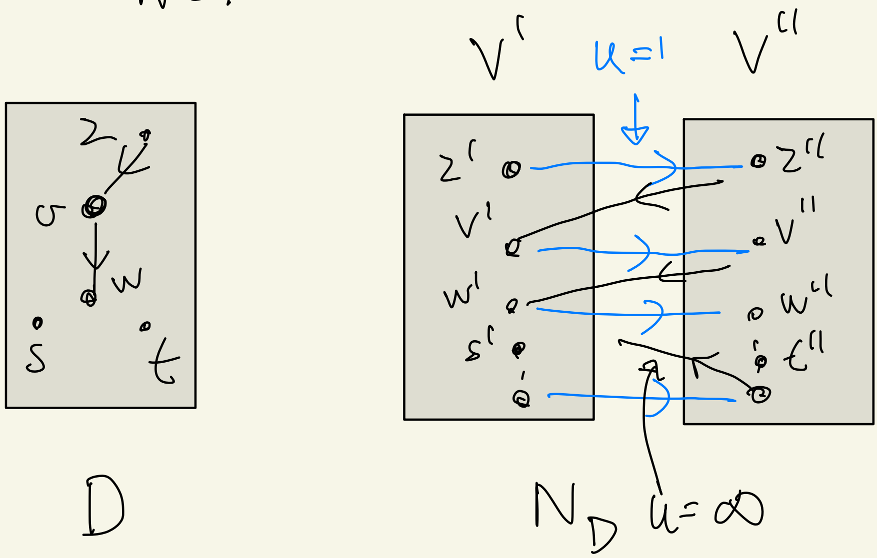
Clear: $\kappa(s,t) \leq \min\{|X| \mid X \text{ is an } (s,t)\text{-sep}\}$

Theorem (Menger) $D=(V,A)$

$s,t \in V$

Suppose $st \notin A$

Then $\kappa(s,t) = \min\{|X| \mid X\ (s,t)\text{-sep}\}$



<u>claim</u>

$\kappa(s,t)$ = max value of an (s'',t')-flow in N_D

Claim

$k(s,t)$ = max value of an (s'',t')-flow in N_D

P: $\le$ Let $P_1, \dots, P_q$ be internally disjoint (s,t)-paths in D and let $P'_1, \dots, P'_q$ be the corresponding paths in N_D. Send 1 unit along each P'_i.

$\ge$ Let x^* be a max (s'',t')-flow (integer valued)

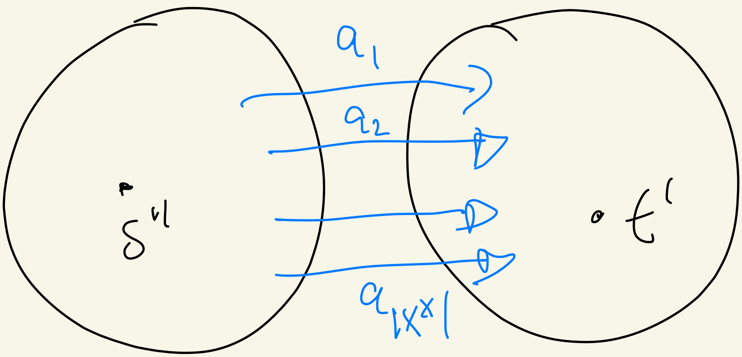
x^* can be decomposed into $|x^*|$ path flows $f(Q_1) \dots f(Q_{|x^*|})$ and some cycle flows

each Q_i, Q_j are internally disj.

So $k(s,t) \ge |x^*|$

$\kappa(s,t) = |X^*| = u(S,\bar{S})$

$=$ # δ ... arcs from S to $\bar{S}$



each $a_i \Leftrightarrow$ a vertex v_i in D

and removing $v_1, v_2, \dots, v_{|X^*|}$

kills all (s,t)-paths in D

So $X = \{v_1, v_2 \dots v_{|X^*|}\}$ is an (s,t)-separator with $\kappa(s,t) = |X|$ $\square$