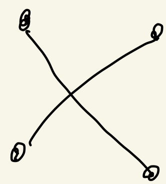
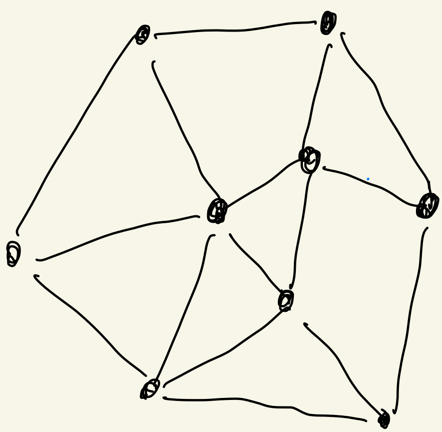
Ahuja 8.4 Flows in planar undirected networks

$G=(V,E)$ is planar if we can draw it in the plane with no edge crossings





f = # faces
m = # edges
n = # vertices

Euler's formula: $f = m - n + 2$

'check' $9 = 16 - 9 + 2$

Property 8.7 $m < 3n$ ($m \le 3n-6$ always)

P: suppose $m \ge 3n$

- Each cycle has at least 3 edges

  so $3f \le 2m$

- $2 = n - m + f$ by formula

So if $m \ge 3n$ we get $n \le \frac{m}{3}$ and

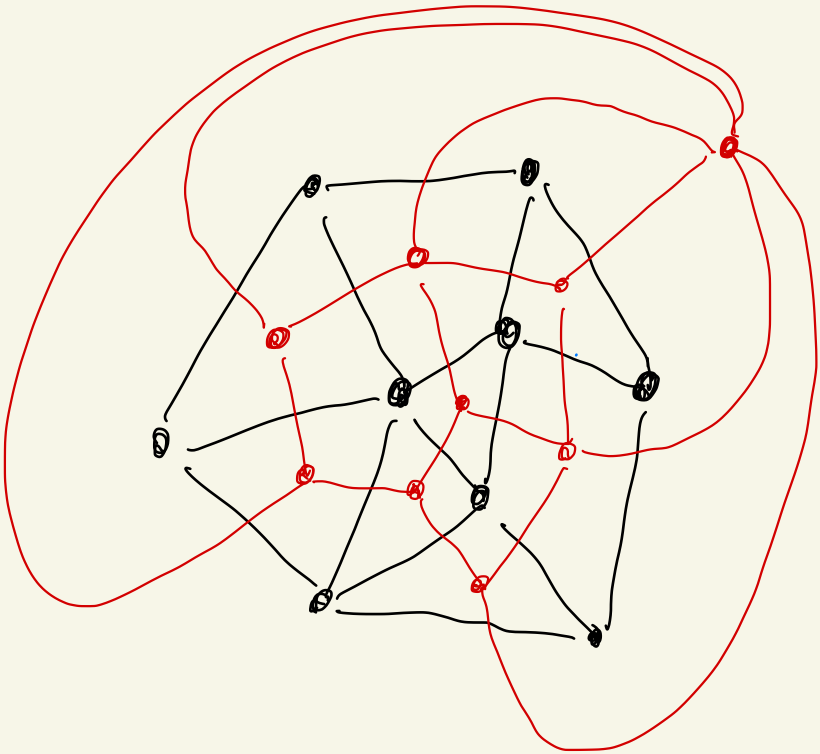
$$2 = n - m + f$$

$$\le \frac{m}{3} - m + \frac{2m}{3}$$

$$= 0$$

↯

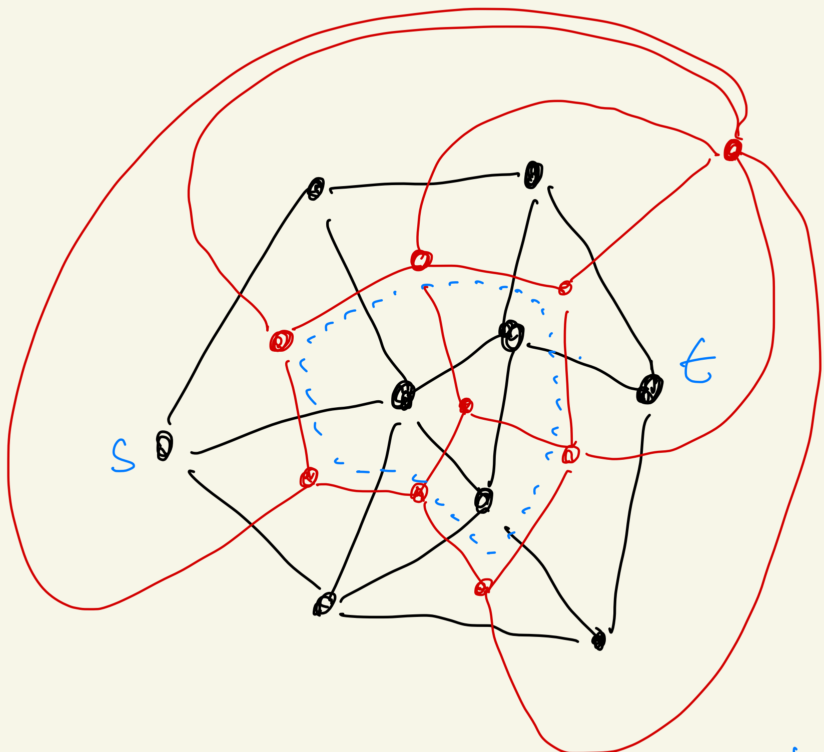
Dual of G: G^D



$(G^D)^D = G$

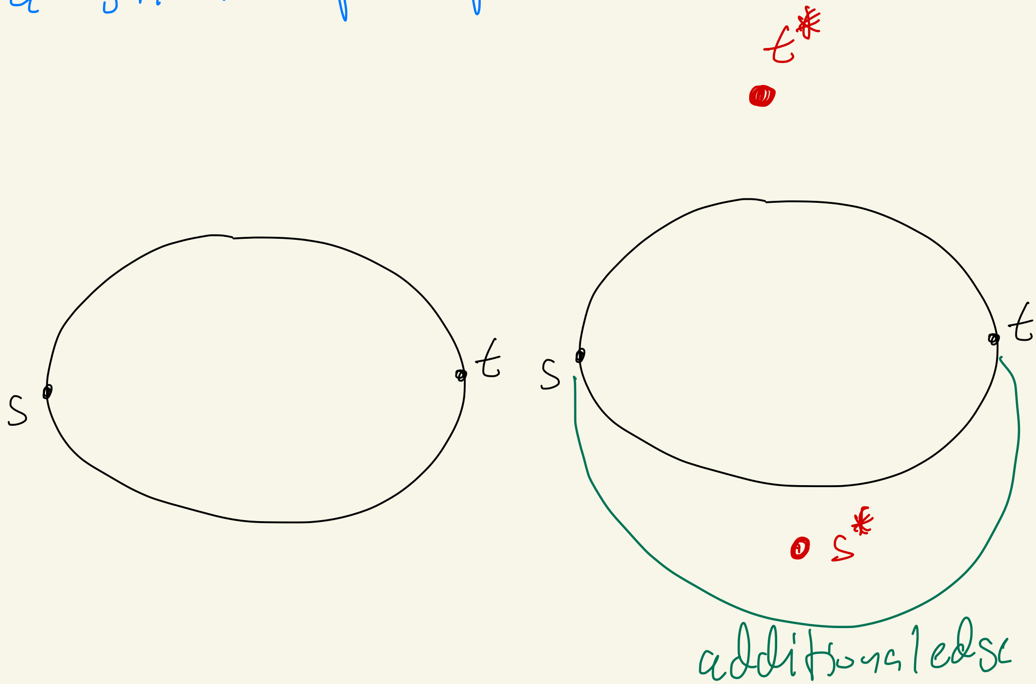
G^D planar

Assumption the source s and the sink t are on the outer boundary

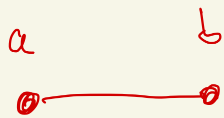


Cycle in dual ⟺ vertex partition (cut) in G

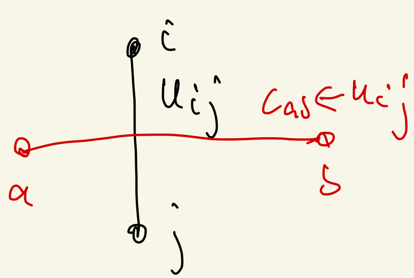
Transforms min(s,t)-cut problem into a shortest path problem:

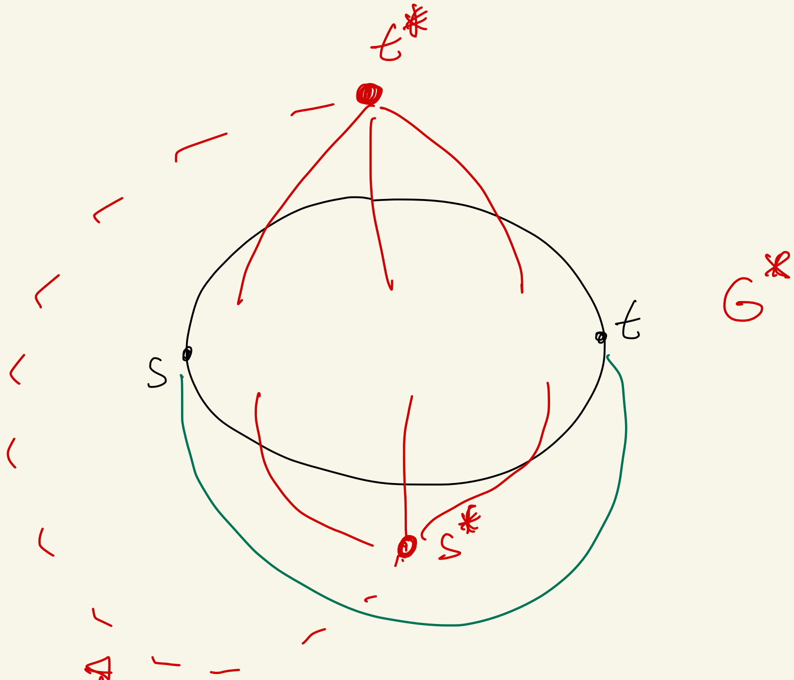


Set cost of edge



in dual equal to the capacity of the edge in G which it crosses:



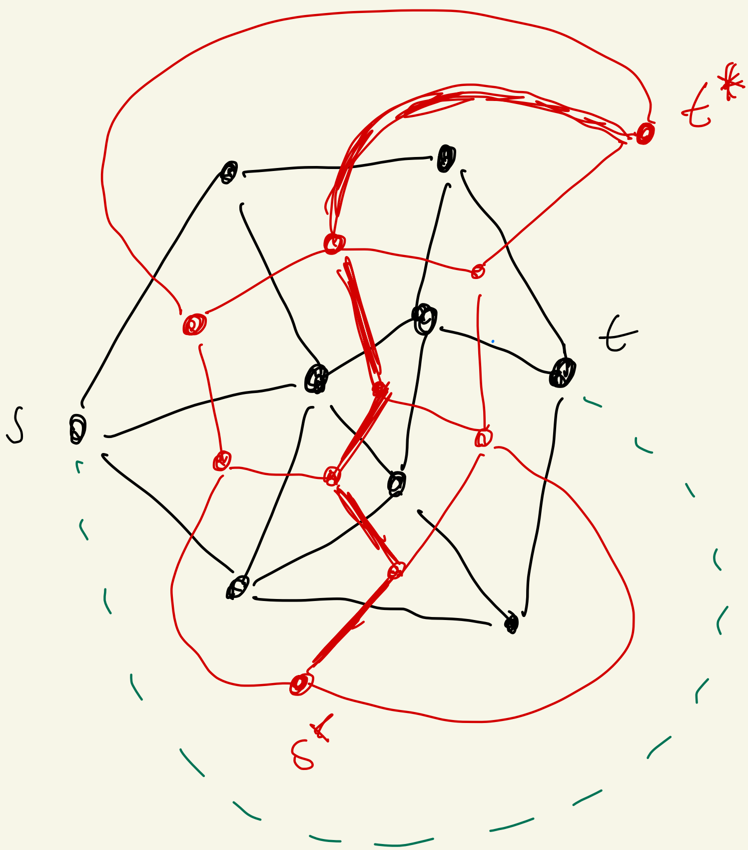


delete this edge from the dual of
G + st

Now there is a 1–1 correspondance
between (s, t)-cuts in G and
(s^*, t^*)-paths in G^* and

$$u(S, \bar{S}) = c(P^*_{s^*t^*})$$

when P^*_{st} is the (s, t)-path in G^D
that corresponds to the cut $(S, \bar{S})$



Each (s^*, t^*)-path in red corresponds to an (s,t)-cut in G

Conclusion: we can find a minimum (s,t)-cut in time $O(n\log n + m\log n) = O(n\log n)$ via Dijkstra in Dual.

obtaining a max flow via distance labels in the dual (G^*)

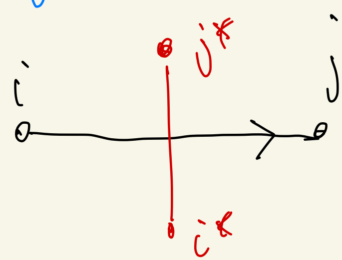
$G^* = (V^*, E^*)$

$d(j^*)$ = length of shortest (s^*, j^*)-path in G^*

(*) $d(j^*) \le d^*(i^*) + c_{i^*j^*} \quad \forall i^*j^* \in E^*$

let $x_{ij} = d(j^*) - d(i^*) \quad \forall ij \in E$

when i^*j^* is the dual edge crossing ij

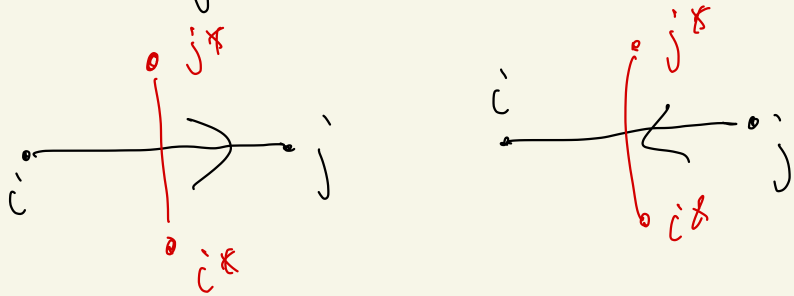


i^* to the right when moving from i to j in G

$$x_{ij} = d(j^*) - d(i^*)$$
$$\le d(i^*) + c_{i^*j^*} - d(i^*)$$
$$= c_{i^*j^*} = u_{ij} \Leftarrow \text{by def of costs in } G^*$$

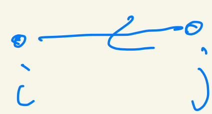
Hence x is feasible and we have

$$x_{ij} = -x_{ji}$$



$$x_{ij} = d(j^*) - d(i^*) \qquad x_{ji} = d(i^*) - d(j^*)$$

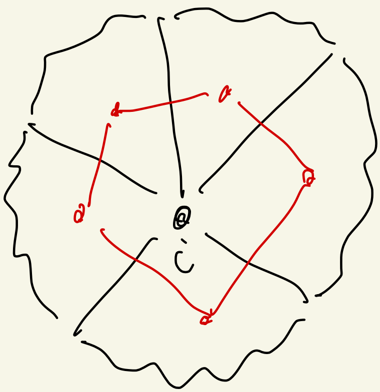
We interpret $x_{ij} < 0$ as $x_{ji} > 0$

and think of ij oriented as 

This way the resulting flow is never negative and $0 \le x_{ij} \le u_{ij}$ holds

Checking that $b_x(i) = 0$ for $i \neq s, t$:

Consider the cut $(i, N-i)$



The edges in G^* corresponding to the edges incident with i form a cycle W^*

so

$$① = \sum_{i^*j^* \in W^*} (d(j^*) - d(i^*))$$

$$= \sum_{ij \in E} x_{ij}$$

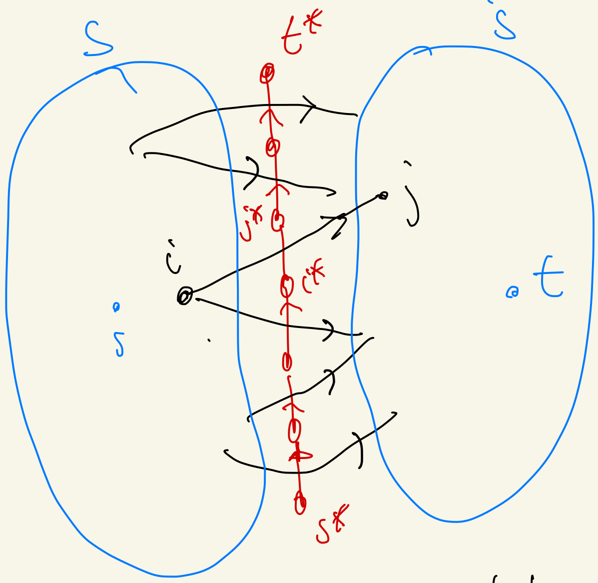
$$= b_x(i)$$

Thus x is an (s,t)-flow.

Maximum:

Let P^* be a shortest (s^*,t^*)-path in G^*

Then $d(j^*) - d(i^*) = c_{i^*j^*} = u_{ij}$ $\forall i^*j^* \in A(P^*)$



Each arc ij across the cut has

$$x_{ij} = d(j^*) - d(i^*) = c_{i^*j^*} = u_{ij}$$

so $(S,\bar{S})$ is a min cut and x is a max flow

Theorem 8.8 A maximum (s,t)-flow in a planar network can be found in time $O(n \log n)$ when s and t are on the boundary