A General Cut

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A general cut

In solving an integer programming problem

$$\begin{array}{lrcl} \max & cx & & \\ & Ax & = & b \\ & x & \geq & 0, x \in Z^n \end{array}$$

one often starts by solving the LP-relaxation. Let us denote the optimal basis of the LP-relaxation B.

- The i'th row in the optimal Simplex tableau now expresses the following equation:

$$x_{B_i} + \sum_{j \notin B} \alpha_{ij} x_j = \beta_i$$

  in which α_{ij} and β_i are the coefficients in the i'th row of the tableau.

- NOW, if all RHS in the Simplex tableau are integer the optimal solution has been found.

If β_i fractional then...

If not, we construct an inequality, which is satiesfied by all solutions to the IP as follows:

- consider a row i, in which β_i is non-integral.
- separate α_{ij} and β_i into their **integral** parts $\lfloor \alpha_{ij} \rfloor$ and $\lfloor \beta_i \rfloor$ and their **fractional** parts $r\alpha_{ij}$ and $r\beta_i$:

$$\alpha_{ij} = \lfloor \alpha_{ij} \rfloor + r\alpha_{ij}, \beta_i = \lfloor \beta_i \rfloor + r\beta_i$$

where $\lfloor \alpha_{ij} \rfloor, \lfloor \beta_i \rfloor \in Z$, $r\alpha_{ij}, r\beta_i \in [0, 1[$.

$$x_{B_i} + \sum_{j \notin B} \alpha_{ij} x_j = \beta_i \Leftrightarrow$$

$$x_{B_i} + \sum_{j \notin B} (\lfloor \alpha_{ij} \rfloor + r\alpha_{ij}) x_j = \lfloor \beta_i \rfloor + r\beta_i$$

Since all x are non-negative, LHS becomes smaller if the fractional parts are discarded in the coefficients of x:

$$x_{B_i} + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j \leq \lfloor \beta_i \rfloor + r\beta_i$$

Now LHS is integral for all integral x. If $r\beta_i$ is discarded in the inequality, it is thus valid for all integral solutions (but not necessarily for non-integral solutions):

$$x_{B_i} + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j \leq \lfloor \beta_i \rfloor$$

The constructed inequality is now subtracted from the considered “row equation” giving an inequality, which is valid for all integer solutions, but not necessarily for non-integral solutions:

$$
\begin{array}{rlcl}
 & x_{B_i} + \sum_{j \notin B} (\lfloor \alpha_{ij} \rfloor + r\alpha_{ij}) x_j & = & \lfloor \beta_i \rfloor + r\beta_i \\
- & (x_{B_i} + \sum_{j \notin B} \lfloor \alpha_{ij} \rfloor x_j & \leq & \lfloor \beta_i \rfloor) \\
\hline
 & \sum_{j \notin B} r\alpha_{ij} x_j & \geq & r\beta_i
\end{array}
$$

- Since $x_j = 0$ for $j \notin B$ the above inequality does not hold for the basic LP-solution.
- So it is a **valid inequality**, which when added to the problem *cuts off* the current LP-solution still leaving all integer solutions in the feasible set.
- This inequality is called a **Gomory-cut**.