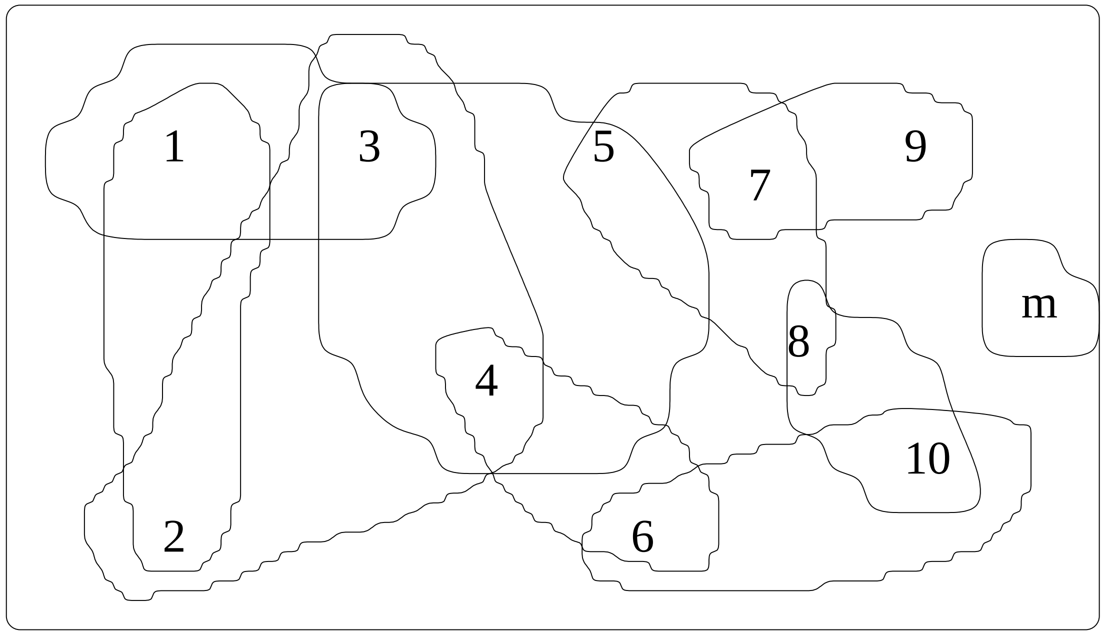
Solving real-life problems

Jesper Larsen
Informatics and Mathematical Modelling
Technical University of Denmark
2800 Kgs. Lyngby
Email: jla@imm.dtu.dk

Let $I = \{1, 2, \ldots, m\}$ and $P = \{P_1, P_2, P_3, \ldots, P_n\}$ where $P_j \subseteq I$.



Now let $J \subseteq \{1, 2, \ldots, n\}$. Furthermore we associate a cost for each P_j. The cost of J is given by $\sum c_j$.

- If $\cup_{j \in J} P_j = I$ J is called a **set cover**.
- If $P_j \cap P_k = \emptyset$ for $i, j \in J$ J is called a **set packing**.
- If J is a set cover **and** a set packing J is called a **set partitioning**

Set covering

$a_{ij} = 1$ if $i \in P_j$ and 0 otherwise, and $x_j = 1$ if $j \in J$ and 0 otherwise.

$$
\begin{array}{lll}
\min & \sum_{j=1}^{n} c_j x_j & \\
\text{st.} & \sum_{j=1}^{n} a_{ij} x_j \geq 1 & i = 1, 2, \ldots, m \\
& x_j = 0, 1 & j = 1, 2, \ldots, n
\end{array}
$$

Set packing

$a_{ij} = 1$ if $i \in P_j$ and 0 otherwise, and $x_j = 1$ if $j \in J$ and 0 otherwise.

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{st.} & \sum_{j=1}^{n} a_{ij} x_j \leq 1 & i = 1, 2, \ldots, m \\ & x_j = 0, 1 & j = 1, 2, \ldots, n \end{array}$$

Set partitioning

$a_{ij} = 1$ if $i \in P_j$ and 0 otherwise, and $x_j = 1$ if $j \in J$ and 0 otherwise.

$$\begin{array}{lll} \min & \sum_{j=1}^{n} c_j x_j & \\ \text{st.} & \sum_{j=1}^{n} a_{ij} x_j = 1 & i = 1, 2, \ldots, m \\ & x_j = 0, 1 & j = 1, 2, \ldots, n \end{array}$$

Notice the structure: All a_{ij}'s are 0 or 1 and the right hand side in the constraints are 1's.

Some models can contain other integers than 1 on the right hand side, and one would typically describe these problems as a **general** set packing/covering/partitioning problem.

Example

Assume $I = \{1, 2, 3, 4, 5, 6\}$, and we have $P_1 = \{1, 2\}$, $P_2 = \{1, 3, 4\}$, $P_3 = \{2, 4, 5\}$, $P_4 = \{3, 5, 6\}$, $P_5 = \{4, 5, 6\}$; $c_1 = 5$, $c_2 = 4$, $c_3 = 6$, $c_4 = 2$ and $c_5 = 4$.

How would the fomulation of the set covering model for these data look like?

A scheduling problem

We have 6 assignments A, B, C, D, E and F, that needs to be carried out. For every assignment we have a start time and a duration (in hours).

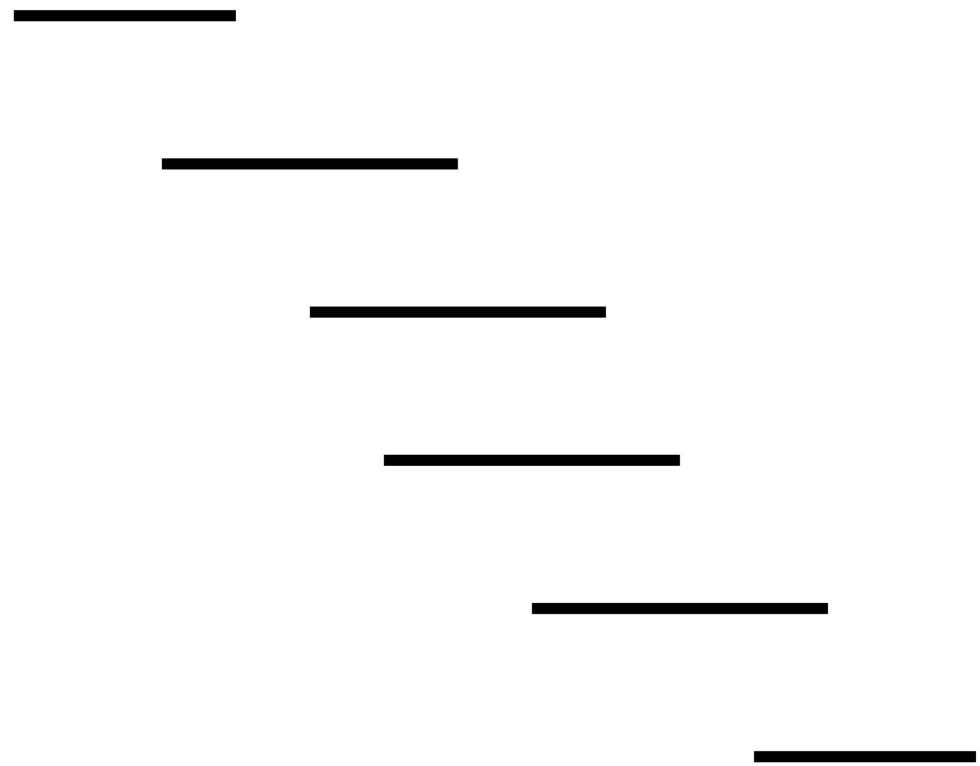
Assignment	A	B	C	D	E	F
Start	0.0	1.0	2.0	2.5	3.5	5.0
Duration	1.5	2.0	2.0	2.0	2.0	1.5

Formulate a set partitioning model that finds the cheapest set of workplans that fullfills all the assignments.

Notice

- A workplan can not consist of assignments that overlap each other.
- The length L of a workplan is equal to the sum of the durations of the assignments of the plan plus 30 minutes for checking in and checking out.
- the cost of a workplan is $\max(4.0, L)$.

	Start	Duration
A	0.0	1.5
B	1.0	2.0
C	2.0	2.0
D	2.5	2.0
E	3.5	2.0
F	5.0	1.5



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minimize 4x1 + 4.5x2 + 7x3 + 5x4 + 7x5 + 6x6 + 7x7 +
4x8 + 5x9 + 6x10 + 4x11 + 5x12 + 4x13 + 4.5x14 + 4x15 +
st
x1+x2+x3+x4+x5+x6+x7=1
x8+x9+x10=1
x2+x3+x11+x12=1
x4+x5+x13+x14=1
x6+x9+x15=1
x3+x5+x7+x10+x12+x14+x16=1

integer
x1 x2 ... x16
end
```

Productionplanning

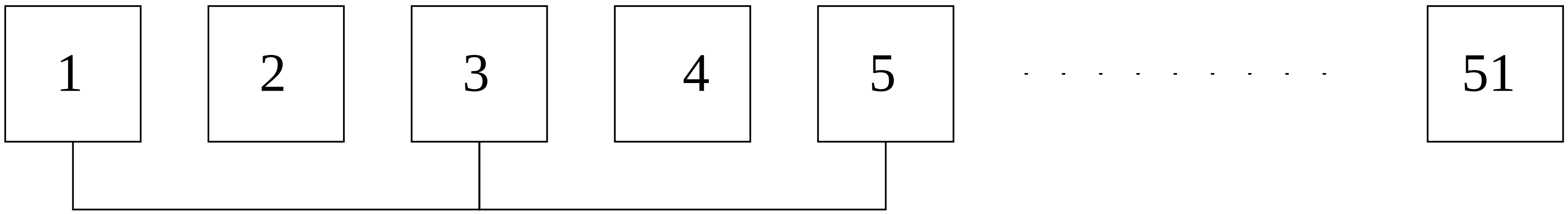
- This is the chemical reaction in a Hall-Heroult cell

$$(2Al_2O_3 + 3C) + \text{ Energy} \rightarrow 4Al + 3CO_2$$

- Every “reduction line” (600 m long) is devided into 4 tapping bays (300 m long) every consisting of 51 cells.
- All active cells in a tapping bay is tapped once a day.

- The purity of aluminium in a cell depends on
  - the age of the cell,
  - the purity of aluminium ore, and
  - the carefullness of the workers.
- The purity in the metal in a cell is determined once a day:
- Possible impurities: Iron, Silicium, Gallium, Nickel, Vanadium

- A “batch” consists of metal from 3 cells.
- The purity of the batch is the average of the purity of contributing cells.
- The purity is a very important factor in the pricing of aluminium.
- Due to the long distances in the tapping bay the cells in a batch are not allowed to be to far from each other.



Spredning = 5−1 = 4

- Given a metal purity for each cell in the tapping bay we want to maximize the value of the metal in our batches.
- Under the condition of:
  - ◇ all cells are tapped ones
  - ◇ there are at most three cells in a batch
  - ◇ the "spread" S in a batch is $\leq S_1$
  - ◇ except for at most C batches where $S_1 < S \leq S_2$

Let $S_1 = 4, C = 0$. Consider 6 cells with the following contents

Cell	Al%	Si%	Fe%
652 (1)	99.87	0.050	0.058
653 (2)	99.95	0.022	0.026
654 (3)	99.94	0.020	0.029
655 (4)	99.93	0.023	0.039
656 (5)	99.78	0.024	0.019
657 (6)	99.93	0.022	0.030

Possible batches with cell 1 (652)

Batch	Cells	Al%	Si%	Fe%	Code	Premium
1	1 2 3	99.920	0.031	0.038	AA190K	100
2	1 2 4	99.917	0.032	0.041	AA190K	100
3	1 2 5	99.867	0.032	0.091	AA185G	15
4	1 3 4	99.913	0.031	0.042	AA190K	100
5	1 3 5	99.863	0.031	0.092	AA185G	15
6	1 4 5	99.860	0.032	0.096	AA1709	0

Set partition representation

Batch	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16		
652	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	=	1
653	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	0	=	1
654	1	0	0	1	1	0	1	1	1	0	0	0	1	1	1	0	=	1
655	0	1	0	1	0	1	1	0	0	1	1	0	1	1	0	1	=	1
656	0	0	1	0	1	1	0	1	0	1	0	1	1	0	1	1	=	1
657	0	0	0	0	0	0	0	0	1	0	1	1	0	1	1	1	=	1
	1	1		1			1		1		1			1	1			
	0	0	1	0	1		8	1	8	1	4	1	1	4	8	1		
	0	0	5	0	5	0	0	5	0	5	0	5	5	0	0	5		

Cell Batching Optimization

$$
\begin{array}{ll}
\max & \sum_{i=1}^{n} p_i x_i \\
\text{st} & \sum_{i=1}^{n} a_{ji} x_i = 1 \quad j = 1, 2, \ldots, m \\
& \sum_{i=1}^{n} s_j x_j \leq C
\end{array}
$$

Where $s_j = 1$ if the spread is larger than S_1 and 0 otherwise.

Strategic manpower planning

- Consider an organisation that is open 7 days a week with 1 shift a day.
- The number of employees needed varies from day to day but is constant on a weekly basis $(b_i, i = 1, 2, \ldots, 7)$.
- All employees must work 5 consecutive days and have two days off.
- Minimize the number of employees and find out when they have to work?