## Basic concepts - LP I

We consider an LP-problem **LP** on standard form:

$$
\begin{aligned}
&max\ cx \\
&Ax = b \\
&x \in R^n_+
\end{aligned}
$$

Stated in detail:

$$
\begin{aligned}
max \quad z = c_1x_1 + c_2x_2 + ... + c_nx_n \\
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\
. \\
. \\
. \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \\
x \geq 0 \quad , \quad i \ = \ 1 \, , ... \ n
\end{aligned}
$$

All problems can be transformed to this form.

## Basic concepts - LP II

- A **solution** to LP satisfies $Ax = b$.
- A **feasible solution** to LP satisfies $Ax = b \ \wedge \ x \geq 0$.
- An **optimal solution** to LP, $x^*$ is a feasible solution satisfying that for any other feasible solution $\overline{x}$
  $cx^* \geq c\overline{x}$
- A **basis** for $A$ is a set of $m$ linearly independent columns from $A$.
- **The basic solution** corresponding to the basis

  $$B = A_{:B} = \{A_{j1}, ..., A_{jm}\}$$

  is the solution obtained from $Ax = b$ by setting $x_j = 0 \ , j \notin \{j_1, ..., j_m\}$. This is unique.
- **A basic solution** $\tilde{x}$ to LP is a solution, for which a basis $B$ exists such that $\tilde{x}$ is the basic solution corresponding to $B$.

## Basic concepts - LP III

Consider now the basis

$$B = \{A_{j1}, ..., A_{jm}\}$$

The variables $x_{j_1}, ..., x_{j_m}$ are called **basic variables**, the other variables ( $x_j = 0$ , $j \notin \{j_1, ..., j_m\}$ are **non-basic variables**.

The basic solution corresponding to $B$ is found by

1. set all non-basic variables to 0 in $Ax = b$.
2. solve the "remaining system:

$$Bx = b \Leftrightarrow x = B^{-1}b$$

3. value ? - insert !

## Solving LP-problems - Algebra I.

Consider the problem

$$\begin{array}{rrcl} \max & \mathrm{c}x & & \\ & \mathrm{A}x & = & \mathrm{b} \\ & x & \geq & 0 \end{array}$$

Suppose that we have a basis $B$, a partitioning of $A$ in a basis-part and a non-basis part $A = (B\ N)$, and a corresponding partitioning of the vector of variables $x$ into $(x_B\ x_N)$. The basic solution corresponding to $B$ is algebraically found as follows::

$$\begin{array}{rrcl} \max & cx & & \\ & Ax & = & b \qquad \mapsto \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{rrcl} \max & c_B x_B + c_N x_N & & \\ & B x_B + N x_N & = & b \qquad \mapsto \\ & x_B,\ x_N & \geq & 0 \end{array}$$

## Solving LP-problems - Algebra II.

Left-multiply with $B^{-1}$ and move terms to the right:

$$\begin{array}{rrcl} \max & c_B x_B + c_N x_N & & \\ & I x_B + B^{-1} N x_N & = & B^{-1} b \quad \mapsto \\ & x_B,\ x_N & \geq & 0 \end{array}$$

$$\begin{array}{rrcl} \max & c_B x_B + c_N x_N & & \\ & x_B & = & B^{-1} b - B^{-1} N x_N \quad \mapsto \\ & x_B,\ x_N & \geq & 0 \end{array}$$

Insert the expression for $x_B$ into the objective fctn:

$$\begin{array}{rrcl} \max & c_B(B^{-1}b \ - B^{-1}N x_N) & + & c_N x_N \\ & x_B & = & B^{-1} b - B^{-1} N x_N \quad \mapsto \\ & x_B,\ x_N & \geq & 0 \end{array}$$

Collect terms:

$$\begin{array}{rrcl} \max & 0x_B + (c_N - c_B B^{-1} N) x_N & + & c_B B^{-1} b \\ & I x_B + B^{-1} N x_N & = & B^{-1} b \\ & x_B,\ x_N & \geq & 0 \end{array}$$

The j'th reduced cost: $\boxed{\overline{c_j} = c_j - (c_B B^{-1} N)_j}$.

What is the contents of the Simplex tableau ?

# Solving LP-problems - Algebra II.

$$\begin{aligned} \max\ cx &= c_B x_B + c_N x_N \\ (B\ \ N)\, x &= b \\ x &\geq 0 \end{aligned}$$

| $c_B$ | $c_N$ | $0$ |
|---|---|---|
| $B$ | $N$ | $b$ |

$$\Downarrow$$

$$\begin{aligned} \max\ cx &= (c_N - c_B B^{-1} N)\, x_N + c_B B^{-1} b \\ x_B &= B^{-1} b - B^{-1} N x_N \\ x_B,\ x_N &\geq 0 \end{aligned}$$

$-c_B B^{-1} b$

| $0$ | $c_N - c_B B^{-1} N$ | |
|---|---|---|
| $\begin{matrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{matrix}$ | $B^{-1} N$ | $B^{-1} b$ |

## Solving an LP - I

Consider the problem

$$
\begin{aligned}
max\ \ 7p + 11q \\
1 \le p \le 8 \\
1 \le q \le 3.5 \\
2p - q \ge 0 \\
p + q \le 9 \\
p, q \ge 0
\end{aligned}
$$

Transform:
$x_1 = p - 1, \quad x_2 = q - 1 \Leftrightarrow p = x_1 + 1, \quad q = x_2 + 1$

$$
\begin{aligned}
max\ \ 7x_1 + 11x_2 \quad + 18 \\
0 \le x_1 \le 7 \\
0 \le x_2 \le 2.5 \\
2x_1 - x_2 \ge -1 \\
x_1 + x_2 \le 7 \\
x_1, x_2 \ge -1
\end{aligned}
$$

## Solving an LP - II

In standard form:

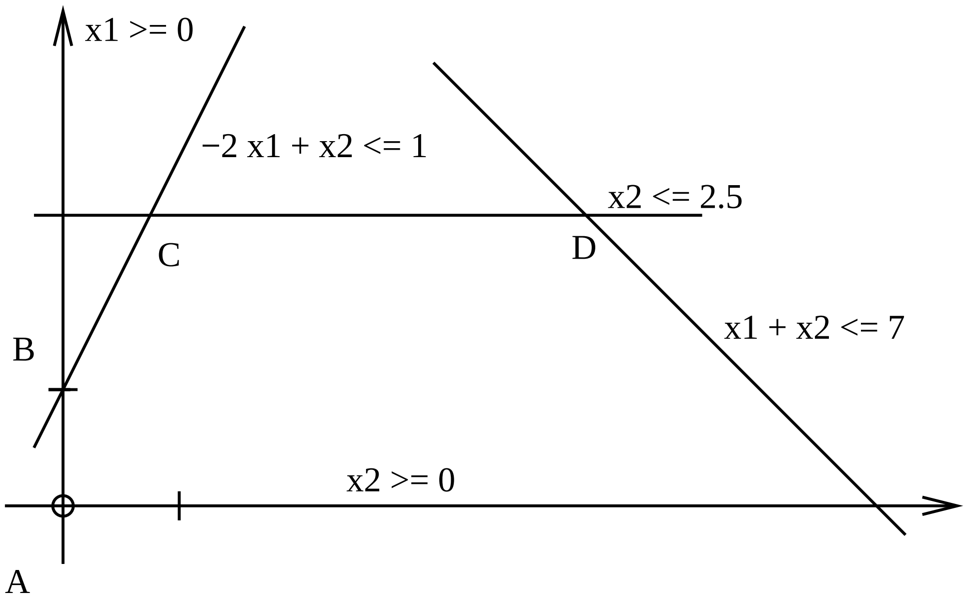
$$\begin{array}{rrrrrrl}
max & 7x_1 + 11x_2 & (+18) & NB! & & & \\
& -2x_1 + x_2 + x_3 & & & & = 1 \\
& x_1 + x_2 & & + x_4 & & = 7 \\
& x_2 + & & & +x_5 & = 2.5 \\
& & & & x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{array}$$

The variables $x_3, x_4, x_5$ are called **slack variables** and are introduced to obtain a system in standard form.

# Solving an LP - III

**Canonical Simplex tableau** wrt. the basis {3,4,5}:

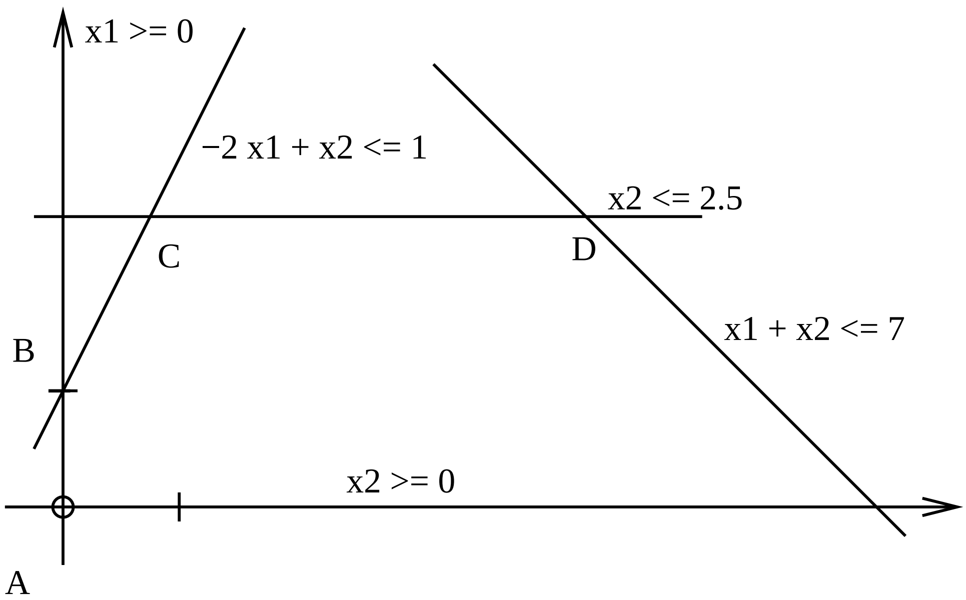
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | 0 |
|---|---|---|---|---|---|---|
| Red. Costs | 7 | 11 | 0 | 0 | 0 | 0 |
| $x_3$ | -2 | 1 | 1 | 0 | 0 | 1 |
| $x_4$ | 1 | 1 | 0 | 1 | 0 | 7 |
| $x_5$ | 0 | 1 | 0 | 0 | 1 | 2.5 |




The basic solution: (0,0,1,7,2.5). Value: 0 (+18).

Optimal: No - increasing $x_1$ or $x_2$ increases the objecctive function. Interplay with $x_3, x_4, x_5$ ?

## Solving an LP - IV



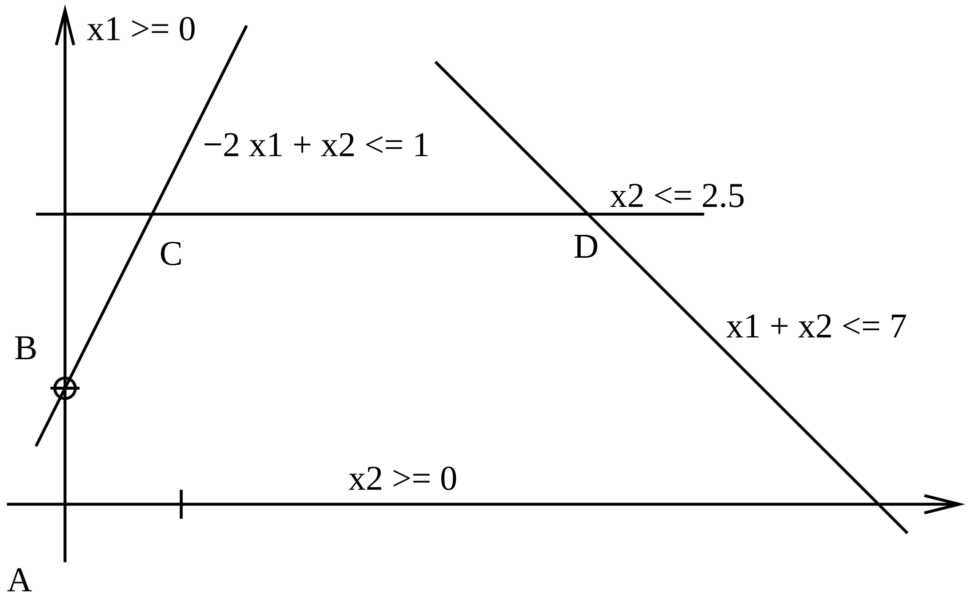

Fix $x_1$ to 0. The the equation system is

$$
\begin{aligned}
max \;\; 11x_2 \\
x_3 = 1 - x_2 \\
x_4 = 7 - x_2 \\
x_5 = 2.5 - x_2 \\
x_2, ..., x_5 \geq 0
\end{aligned}
$$

$x_3, x_4, x_5$ all decrease when $x_2$ increases.

Increase $x_2$ as much as possible. Bounds: all variables must stay non-negative. $x_3$ **sets the bound** - $x_2$ can be increased to 1. Find Simplex tableau wrt. the basis {2,4,5}.

## Solving an LP - V

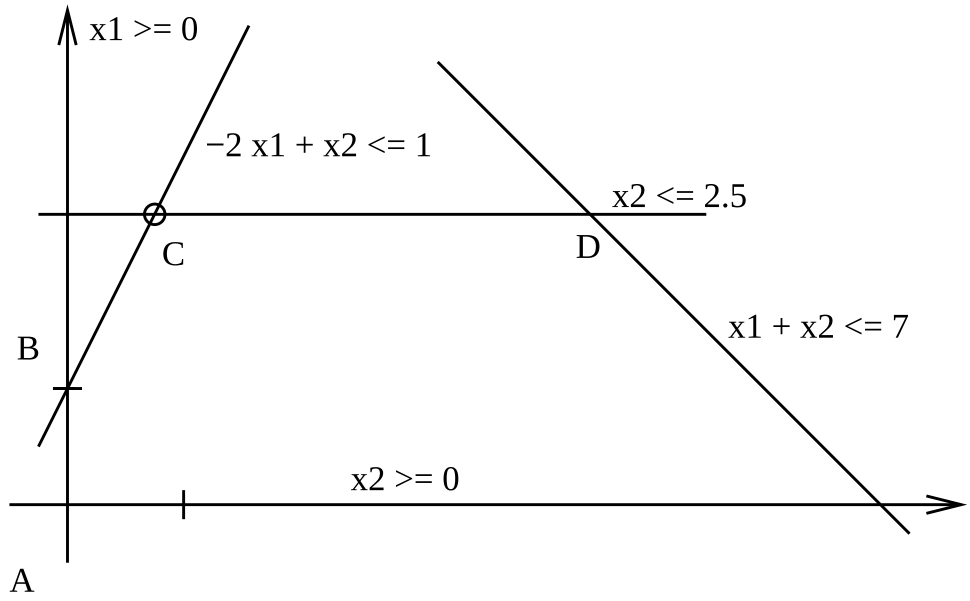



| Red. Costs | 7 | 11 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|
| $x_3$ | -2 | **1** | 1 | 0 | 0 | 1 |
| $x_4$ | 1 | 1 | 0 | 1 | 0 | 7 |
| $x_5$ | 0 | 1 | 0 | 0 | 1 | 2.5 |

| Red. Costs | 29 | 0 | -11 | 0 | 0 | -11 |
|---|---|---|---|---|---|---|
| $x_2$ | -2 | 1 | 1 | 0 | 0 | 1 |
| $x_4$ | 3 | 0 | -1 | 1 | 0 | 6 |
| $x_5$ | 2 | 0 | -1 | 0 | 1 | 1.5 |

Optimal solution ? No - increase $x_1$

## Solving an LP - VI

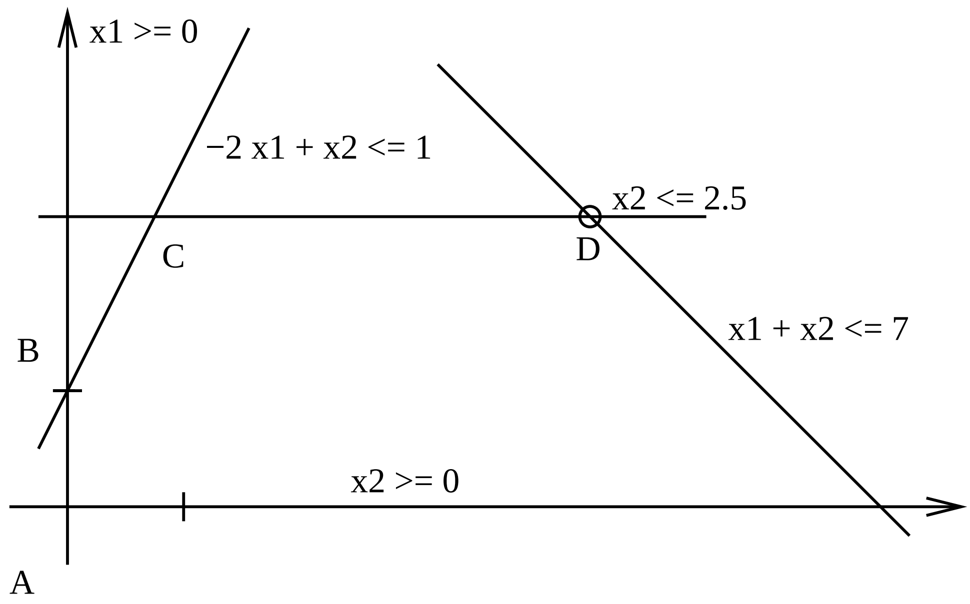



| Red. Costs | 29 | 0 | -11 | 0 | 0 | -11 |
|---|---|---|---|---|---|---|
| $x_2$ | -2 | 1 | 1 | 0 | 0 | 1 |
| $x_4$ | 3 | 0 | -1 | 1 | 0 | 6 |
| $x_5$ | **2** | 0 | -1 | 0 | 1 | 1.5 |

| Red. Costs | 0 | 0 | 7/2 | 0 | -29/2 | -131/4 |
|---|---|---|---|---|---|---|
| $x_3$ | 0 | 1 | 0 | 0 | 1 | 5/2 |
| $x_4$ | 0 | 0 | 1/2 | 1 | -3/2 | 15/4 |
| $x_1$ | 1 | 0 | -1/2 | 0 | 1/2 | 3/4 |

Optimal solution ? No - increase $x_3$

## Solving an LP - VII




| Red. Costs | 0 | 0 | 7/2 | 0 | -29/2 | -131/4 |
|---|---|---|---|---|---|---|
| $x_3$ | 0 | 1 | 0 | 0 | 1 | 5/2 |
| $x_4$ | 0 | 0 | **1/2** | 1 | -3/2 | 15/4 |
| $x_1$ | 1 | 0 | -1/2 | 0 | 1/2 | 3/4 |

| Red. Costs | 0 | 0 | 0 | -7 | -4 | -59 |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 5/2 |
| $x_3$ | 0 | 0 | 1 | 2 | -3 | 15/2 |
| $x_1$ | 1 | 0 | 0 | 1 | -1 | 9/2 |

Optimal solution ? Yes ! No variables can be increased without decreasing the objective function.

## The Simplex Algorithm for LP I

The procedure just completed is called the Simplex Algorithm and can be described as follows:

**Input:** A maximization-LP-problem in canonical form with respect to a basis, i.e. such that the columns of the basic variables are unit vectors, and that each basic variable has 0 as coefficient in the objective function.

- optimal := unbounded := "no" ;
- while optimal = "no" and unbounded = "no" do
  - if $\overline{c_j} \leq 0$ for all $j$ then optimal := "yes" else
  - choose $s$ with $\overline{c_s} > 0$ (often largest positive); *)
    **j is pivot column**
  - if $\overline{a_{is}} \leq 0$ for all $i$ then unbounded := "yes" else
    - find $q = min_{i:\overline{a_{is}}>0} \; \{\overline{b_i}/\overline{a_{is}}\} \quad = \quad \overline{b_r}/\overline{a_{rs}}$
      **r is pivot row**, $q$ is the increase in objective function value from the current basic solution to new (to be constructed).
    - pivot on $\overline{a_{rs}}$

*): If the problem in question is a minimization problem, then "$\overline{c_j} > 0$ (often largest positive)" is to be "$\overline{c_j} < 0$ (often smallest negative)".

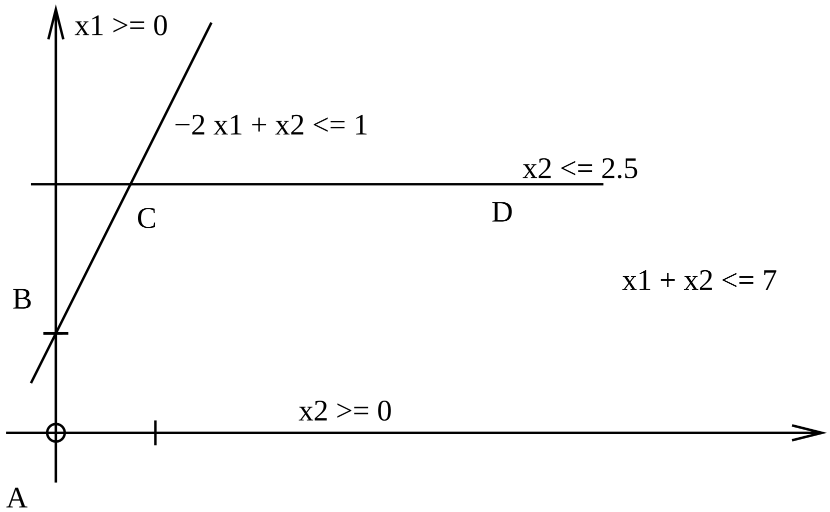
Pivot ? - Exact description ?

## The Simplex Algorithm for LP - pivoting

The Pivot operation on $\overline{a_{rs}}$:

1. Divide row $r$ with $\overline{a_{rs}}$ to produce tableau with a 1 in row $r$, column $s$
   $\overline{a_{rj}} := \overline{a_{rj}}/\overline{a_{rs}}, \quad j \in \{1, ..., n\}$
2. For each other row $p$ (including the "objective function row"), subtract a multiplum of row $r$ such that $\overline{a_{ps}}$ becomes 0 in the new tableau:
   - $\overline{a_{pj}} := \overline{a_{pj}} - (\overline{a_{rj}}/\overline{a_{rs}}) \cdot \overline{a_{ps}} \quad j \in \{1, ..., n\}$
   - $\overline{b_p} := \overline{b_p} - (\overline{b_r}/\overline{a_{rs}}) \cdot \overline{a_{ps}}$

## Removing a constraint




| Red. Costs | 7 | 11 | 0 | 0 | 0 |
|---|---|---|---|---|---|
| $x_3$ | -2 | **1** | 1 | 0 | 1 |
| $x_4$ | 0 | 1 | 0 | 1 | 2.5 |
| Red. Costs | 29 | 0 | -11 | 0 | -11 |
| $x_2$ | -2 | 1 | 1 | 0 | 1 |
| $x_4$ | **2** | 0 | -1 | 1 | 1.5 |
| Red. Costs | 0 | 0 | 7/2 | 0 | -131/4 |
| $x_2$ | 0 | 1 | 0 | 0 | 5/2 |
| $x_1$ | 1 | 0 | -1/2 | 0 | 3/4 |

The problem is **unbounded** - $x_3$ can be increased infinitely, increasing the objective function all the way ...

## Optimality argument I

- The Simplex tableau for the basis {1,2,3} is just **another representation** of the equations and inequalities of the original problem.
- For any feasible $x$ : $x \in \{x | Ax = b \ , x \geq 0\}$ it holds that

$$\boxed{cx = \overline{c}x + c_B B^{-1} b}$$

where $\overline{c}$ are the reduced costs corresponding to the basis $B$.

Example:

| Red. Costs | $\overline{c_1}$ | $\overline{c_2}$ | $\overline{c_3}$ | $\overline{c_4}$ | $\overline{c_5}$ | - $c_B B^{-1} b$ |
|---|---|---|---|---|---|---|
| A | 7 | 11 | 0 | 0 | 0 | 0 |
| C | 0 | 0 | 7/2 | 0 | -29/2 | -131/4 |
| D | 0 | 0 | 0 | -7 | -4 | -59 |

| $x_1, ..., x_5$ | A-vers. | D-vers. |
|---|---|---|
| (0,0,1,7,5/2) | 0 | $7 \cdot -7 + 5/2 \cdot -4 + 59$ |
| (0,1,0,6,3/2) | $11 \cdot 1$ | $6 \cdot -7 + 3/2 \cdot -4 + 59$ |
| (9/2,5/2,15/2,0,0) | $7 \cdot 9/2 + 11 \cdot 5/2$ | 59 |

## Optimality argument II

- If for the basis $B$ it holds that $\boxed{\bar{c} \leq 0}$, then $B$ is optimal because:

$$(1*) \quad cx = \bar{c}x + c_B B^{-1} b = \overline{c_N} x_N + c_B B^{-1} b$$

for all $x \in S$ (since $\overline{c_B} = 0$) and

$$(2*) \quad \overline{c_N} x_N \leq 0 \quad since \quad \overline{c_N} \leq 0, \quad x \geq 0$$

- The basic solution for $B$ has all non-basic variables equal 0 and hence have the value

$$c_B B^{-1} b$$

which by (1*) and (2*) is the best possible since

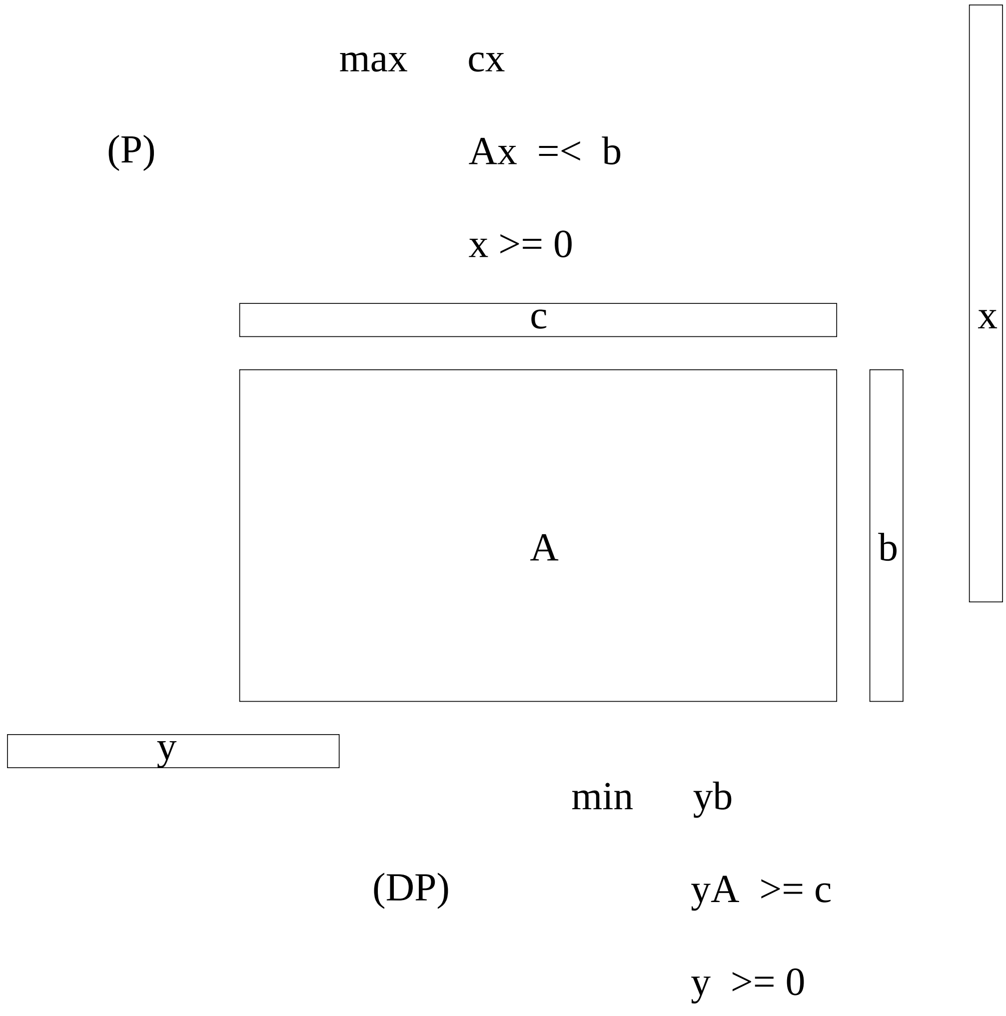
$$(1*) \wedge (2*) \Rightarrow cx = \overline{c_N} x_N + c_B B^{-1} b \leq 0 + c_B B^{-1} b \ , x \in S$$

Convergence ?

Can be shown easily if all basic solutions are different (the problem is **non-degenerate**) - otherwise a mechanism to prevent generating the same basic solution twice during the iterations is necessary.

## Structure of pairs of LP-problems.

LP-problems come in "pairs" – an LP-problem and its dual. The structure of such pairs are illustrated below:




**The Strong Duality Theorem:** If (P) and (DP) both have **feasible** solutions, then both have **optimal** solutions $\overline{x}$ resp. $\overline{y}$, and the optimum values are equal: $\boxed{\mathbf{c\overline{x} = \overline{y}b}}$

## Pairs of primal and Dual LPs.

The connection between the structure of an LP-problem and its dual can be described through the following table. The dual of the dual to a given problem is the problem itself. Equivalent forms of a primal problem have equivalent dual problems.

| Primal: Min | | Dual: Max | |
|---|---|---|---|
| a) i'th constraint | $\geq$ | i'th variable | $\geq 0$ |
| b) ” | $\leq$ | ” | $\leq 0$ |
| c) ” | $=$ | ” | free |
| d) j'th variable | $\geq 0$ | j'th constraint | $\leq$ |
| e) ” | $\leq 0$ | ” | $\geq$ |
| d) ” | free | ” | $=$ |

| Primal: Max | | Dual: Min | |
|---|---|---|---|
| a) i'th constraint | $\geq$ | i'te variable | $\leq 0$ |
| b) ” | $\leq$ | ” | $\geq 0$ |
| c) ” | $=$ | ” | free |
| d) j'th variable | $\geq 0$ | j'th constraint | $\geq$ |
| e) ” | $\leq 0$ | ” | $\leq$ |
| d) ” | free | ” | $=$ |

## Optimality conditions.

**The Complementary Slackness Theorem** is concerned with necessary and sufficient conditions for optimality of feasible solutions $x$ and $y$ to a pair of dual LP-problems:

We consider a pair of dual LP-problems, P and DP:

$$
\begin{array}{llll}
 & \text{(P)} & & \\
\max & \text{cx} & & \\
 & \text{Ax} & \leq & \text{b} \\
 & \text{x} & \geq & 0
\end{array}
\qquad\qquad
\begin{array}{llll}
 & \text{(DP)} & & \\
\min & \text{yb} & & \\
 & \text{yA} & \geq & \text{c} \\
 & \text{y} & \geq & 0
\end{array}
$$

and a pair $\overline{x}, \overline{y}$ of *feasible* solutions to P resp. DP. $\overline{x}$ and $\overline{y}$ are **optimal** solution to P resp. DP *if and only if*:

$$\boxed{(\mathbf{b} - \mathbf{A}\overline{\mathbf{x}}) \cdot \overline{\mathbf{y}} = \mathbf{0} \quad \wedge \quad (\overline{\mathbf{y}}\mathbf{A} - \mathbf{c}) \cdot \overline{\mathbf{x}} = \mathbf{0}}$$

The conditions spelled out are:

$$\forall\ i\ :\ \overline{y}_i(b_i - A_{i\cdot}\overline{x}) = 0$$

$$\forall\ j\ :\ (\overline{y}A_{\cdot j} - c_j)\overline{x}_j = 0$$

## Optimality conditions - revised.

We consider an LP-problem, P:

$$
\begin{array}{lrcl}
\max & \mathrm{cx} & & \\
 & A_1\mathrm{x} & \geq & b_1 \\
 & A_2\mathrm{x} & = & b_2 \\
 & \mathrm{x} & \text{free} &
\end{array}
$$

A given $x^o$ is an optimal solution to P *if and only if*:
1) $x^o$ is a feasible solution to P , and
2) there exists a vector $(y_1, y_2)$ satisfying the following:

$$
\begin{array}{rcll}
(y_1, y_2)\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} & = & \mathrm{c} & (1) \\
y_1(b_1 - A_1x^o) & = & 0 & (2) \\
y_2(b_2 - A_2x^o) & = & 0 & (3) \\
((y_1, y_2)\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} - c)x^o & = & 0 & (4) \\
y_1 & \leq & 0 & (5)
\end{array}
$$

(1)+(5): $(y_1, y_2)$ is feasible for the dual problem of P.
(2)+(3)+(4): the compl. slackness thm. holds for $x^o, (y_1, y_2)$. (3) holds due to the feasibility of $x^o$, and (4) holds due to (1). Thus (3) and (4) are superfluous.

## Optimality conditions - Geometry I

We consider now an LP-problem, P:

$$\begin{array}{rcl} \max & \mathrm{cx} & \\ A\mathrm{x} & \geq & b \\ \mathrm{x} & \text{free} & \end{array}$$

A given $x^o$ is an optimal solution to P *if and only if*:
1) $x^o$ is a feasible solution to P, and
2) there exists a vector $(y)$ satisfying the following:

$$\begin{array}{rcll} yA & = & \mathrm{c} & (1) \\ y(b - Ax^o) & = & 0 & (2) \\ y & \leq & 0 & (3) \end{array}$$

What is the geometric interpretation of (1), (2), and (3) ?

The vector $c$ is the **gradient** of the objective function $cx$, i.e. that direction for a move in $R^n$, for which the objective function increases most rapidly.

## Optimality conditions - Geometry II

The feasible region $S$ for P, $\{x|Ax \geq b\}$, is defined by the functions
$a_{11}x_1 + ... + a_{1n}x_n - b_1 \geq 0$ , ... , $a_{m1}x_1 + ... + a_{mn}x_n - b_m \geq 0$.
The gradients $(a_{11}$ , ... , $a_{1n})$ , ... , $(a_{m1}$ , ... , $a_{mn})$ for these functions point *into* $S$. Condition (1) and (3) state, that at the optimal point of $S$ - an extreme point, $Q$ - the gradient of the objective function must be a **non-positive linear combination** of the gradients of the constraints. For $Q$ (which corresponds to a basic feasible solution), (2) states that the linear combination must be constructed using only the gradients of those constraints, which are **binding** at $Q$: