Exercise 8.5

We will use Gomory's cutting plane algorithm. We add a surplus variable s_4 to the constraint and multiply the objective by -1, to obtain a maximization problem. The simplex tableau for the problem is then:

max	x_1	x_2	x_3	s_4	
-1	5	9	23		0
	-20	-35	-95	1	-319

All coefficients of the objective are positive and the right hand side of the constraint is negative, hence the solution is dual feasible but not optimal. Thus we can use dual-simplex to find the optimal solution. We do the coefficient test (row marked "Test") and find the least negative number ($\frac{23}{-95}$).

max	x_1	x_2	x_3	s_4	
-1	5	9	23		0
	-20	-35	-95	1	-319
Test	$\frac{5}{-20}$	$\frac{9}{-35}$	$\frac{23}{-95}$		

Pivoting the circled element gives the optimal simplex tableau:

max	x_1	x_2	x_3	s_4	
-1	$15/95$	$50/95$		$23/95$	$-7337/95 \approx -77,23$
	$20/95$	$35/95$	1	$-1/95$	$319/95$

The right hand side of the row is not integer (and hence neither is x_3), so we use this row to get a Gomory cut. The Gomory cut obtained is:

$$\frac{20}{95}x_1 + \frac{35}{95}x_2 + \frac{94}{95}s_4 \geq \frac{34}{95}$$
$$\Rightarrow -\frac{20}{95}x_1 - \frac{35}{95}x_2 - \frac{94}{95}s_4 + s_5 = -\frac{34}{95}$$

Adding this row give us the following tableau. Again we need dual-simplex to reoptimze so the coefficient test is shown including the selected pivot element.

max	x_1	x_2	x_3	s_4	s_5	
-1	$15/95$	$50/95$		$23/95$		$-7337/95 \approx -77,23$
	$20/95$	$35/95$	1	$-1/95$		$319/95$
	$-20/95$	$-35/95$		$-94/95$	1	$-34/95$
Test	$-\frac{15}{-20}$	$-\frac{50}{35}$		$-\frac{23}{94}$		

Carrying out the pivoting gives the following optimal tableau.

max	x_1	x_2	x_3	s_4	s_5	
-1	$10/94$	$41/94$			$23/94$	$-7268/94 \approx -77,32$
	$20/94$	$35/94$	1		$-1/94$	$316/94$
	$20/94$	$35/94$		1	$-95/94$	$34/94$

We not that the original objective z is increasing $(77,23 \rightarrow 77,32)$ thus the lower bound on the solution value gets better - as expected. This process can be continued while improving the lower bound on the solution. At this particular instance any of the two rows can be selected to generate a Gomory cut, since both have right hand sides. The rate of improvement is in general, however, very slow.

Exercise 8.8

Denote the nodes in the odd hole with $H = \{1, 2, \ldots, k\}$ (k odd) such that the edges of the odd hole are 12, 23 and $k1$. For all edges ij it holds that $x_i + x_j \leq 1$. Add all inequalities corresponding to edges in the odd hole and obtain:

$$
\begin{aligned}
2x_1 + 2x_2 + \ldots 2x_k &\leq |H| \\
\Rightarrow x_1 + x_2 + \ldots x_k &\leq \frac{|H|}{2} = \frac{|H|-1}{2} + \frac{1}{2} \\
\Rightarrow x_1 + x_2 + \ldots x_k &\leq \frac{|H|-1}{2}
\end{aligned}
$$

The last $\Rightarrow$ follows from the fact that since $|H|$ is odd $\frac{|H|-1}{2}$ is integer and accordingly all variables on the left hand side are integers, hence $\frac{1}{2}$ can be left out.

Exercise 8.15

(i)
For $u = 1/4$, the following inequality is obtained.

$$
\begin{gathered}
\frac{3}{4}x_1 - x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_4 + \frac{1}{4}x_5 \leq -\frac{1}{2} \\
\Rightarrow -x_2 - x_4 \leq -1 \Rightarrow x_2 + x_4 \geq 1
\end{gathered}
$$

Remember that implicitly we have an upper bound on the variables (binary variables) thus $x_i \leq 1$ are valid inequalities.

Taking 1 time the original constraint and 1 time $x_2 \leq 1$, the following constraint is obtained:

$$3x_1 + (1-4)x_2 + 2x_3 - 3x_4 + x_5 \leq 1 - 2$$

Multiplying through by 1/3 for this constraint we get:

$$x_1 - x_2 + \frac{2}{3}x_3 - x_4 + \frac{1}{3}x_5 \leq -\frac{1}{3}$$
$$\Rightarrow x_1 - x_2 - x_4 \leq -1$$

Adding $x_4 \leq 1$ the desired inequality is obtained:

$$x_1 - x_2 + (1-1)x_4 \leq 1 - 1 \Rightarrow x_1 - x_2 \leq 0 \Rightarrow x_1 \leq x_2$$

(ii)
u_{ij} corresponds to the constraint $x_i + x_j \leq 1$. Set $u_{12} = u_{13} = u_{23} = 1/2$ and remaining u's $= 0$. This gives:

$$\frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_1 + x_3) + \frac{1}{2}(x_2 + x_3) \leq 3 \cdot \frac{1}{2}$$
$$\Rightarrow x_1 + x_2 + x_3 \leq 1$$

Correspondingly the following inequalities are valid $x_1 + x_2 + x_4 \leq 1$, $x_1 + x_3 + x_4 \leq 1$ and $x_2 + x_3 + x_4 \leq 1$. Letting $u_{123} = u_{124} = u_{134} = u_{234} = 1/3$ and remaining u's $= 0$, the desired inequality $(x_1 + x_2 + x_3 + x_4 \leq 1)$ is obtained.