Exercise 10.1

See example 10.1 page 169. This example contains a maximization version of UFL and c is profit. Note that in this exercise, c contains cost *not* profit. We use the formulation given on page 13:

$$\min \sum_{i\in M}\sum_{j\in N} c_{ij}x_{ij} + \sum_{j\in N} f_j y_j \tag{1}$$
$$\sum_{j\in N} x_{ij} = 1 \text{ for } i \in M \tag{2}$$
$$x_{ij} \leq y_j \text{ for } i \in M, j \in N \tag{3}$$
$$x_{ij} \geq 0 \text{ for } i \in M \tag{4}$$
$$j \in N, y_j \in \{0,1\} \text{ for } j \in N \tag{5}$$

We dualize the demand constraints (2) and get:

$$z(u) = \min \sum_{i\in M}\sum_{j\in N} c_{ij}x_{ij} + \sum_{j\in N} f_j y_j - \sum_{i\in M} u_i(\sum_{j\in N} x_{ij} - 1) \tag{6}$$
$$x_{ij} \leq y_j \text{ for } i \in M, j \in N \tag{7}$$
$$x_{ij} \geq 0 \text{ for } i \in M \tag{8}$$
$$j \in N, y_j \in \{0,1\} \text{ for } j \in N \tag{9}$$

Rewriting $z(u)$ we get:

$$z(u) = \min \sum_{i\in M}\sum_{j\in N}(c_{ij} - u_i)x_{ij} + \sum_{j\in N} f_j y_j + \sum_{i\in M} u_i \tag{10}$$
$$= \min \sum_{j\in N} z_j(u) + \sum_{i\in M} u_i \tag{11}$$

with $z_j(u) = \sum_{i\in M}(c_{ij} - u_i)x_{ij} + f_j y_j$. This can be decomposed into N subproblems (i.e. solved independently for each N).

Each of these N subproblems can be solved by inspection using: $\min z_j(u) = \min\{0, \sum_{i\in M} \min[c_{ij} - u_i, 0] + f_j\}$.

$f = (4, 8, 11, 7, 5)$ and $u = (5, 6, 3, 2, 6, 4)$ and we need the revised cost-matrix:

$$c_{ij} - u_i = \begin{pmatrix} 1 & -3 & -4 & -2 & 0 \\ -2 & 4 & -4 & 0 & -5 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -1 & 2 \\ -5 & 2 & 0 & -4 & -1 \\ -1 & -2 & 0 & 4 & -3 \end{pmatrix}$$

For the first column $j = 1$: $z_1(u) = \min\{0, 0-2+0+0-5-1+f_1\} = \min\{0, -8+4\} = -4$. This corresponds to setting $y_1 = 1$ and $x_{21} = x_{51} = x_{61} = 1$, $x_{11} = 0$ and x_{31} and x_{41} can be selected arbitrarily. This is done for all columns obtaining the following values: $z_2(u) = 0$, $z_3(u) = 0$, $z_4(u) = -2$ and $z_5(u) = -4$. The values of the decision variables are (variables not mentioned are zero) $y_1 = y_4 = y_5 = x_{21} = x_{51} = x_{61} = x_{14} = x_{34} = x_{44} = x_{54} = x_{25} = x_{55} = x_{65} = 1$.

The lower bound is then obtained: $z(u) = \sum_{j\in N} z_j(u) + \sum_{i\in M} u_i = (-4+0+0-2-4) + (5+6+3+2+6+4) = 16$.

Modifying the solution it can be seen that opening facility 1 and 4 can cover all clients with negative revised costs. Thus there is probably no gain from opening facility 5. Now it is a matter of, for each client, select the lowest cost facility to serve client. The solution obtained is: $y_1 = y_4 = 1$ and $= x_{14} = x_{21} = x_{34} = x_{44} = x_{51} = x_{61} = 1$. The solution value is $4+7+3+4+1+1+1+3 = 24$ and since it is a feasible solution it is an upper bound and it is at most $24-16 = 8$ units from optimal.

Exercise 10.5

Dualizing the budget constraint gives the following problem:

$$\begin{aligned} z(u) &= \max 10y_1 + 4y_2 + 14y_3 + u(4 - 3y_1 - y_2 - 4y_3) \\ &= \max(10-3u)y_1 + (4-u)y_2 + (14-4u)y_3 + 4u \\ &\text{Subject to: } u \geq 0, y \in B^3 \end{aligned}$$

Or stated in another way:

$$z(u) = \max\{0, 10-3u\} + \max\{0, 4-u\} + \max\{0, 14-4u\} + 4u$$

This is a piecewise linear function, so its minimum can be found by considering all breaks and endpoints of the function, i.e. $u = 0$, $u = 3\frac{1}{3}$, $u = 4$, $u = 3\frac{1}{2}$ and $u \to \infty$. The values are: $z(0) = 28$, $z(3\frac{1}{3}) = 14\frac{2}{3}$, $z(4) = 16$, $z(3\frac{1}{2}) = 14\frac{1}{2}$ and $\lim_{u\to\infty} z(u) = \infty$. Thus the optimal(minimum) value of u is $3\frac{1}{2}$ with the lower bound value $14\frac{1}{2}$.

The step of the subgradient algorithm is given by:

$$u^{k+1} = \max\{u_k - \mu_k(4 - 3y_1^* - y_2^* - 4y_3^*), 0\}$$

Initial parameters are: $u^0 = 0$, $\mu_0 = 1$ and $\rho = 1/2$.

The following table shows the first iterations of a run of the subgradient algorithm.

k	u	μ
0	0	1
1	$\max\{0 - 1(4 - 3 - 1 - 4), 0\} = 4$	1/2
2	$\max\{4 - 1/2(4 - 0 - 0 - 0), 0\} = 2$	1/4
3	$\max\{2 - 1/4(4 - 3 - 1 - 4), 0\} = 3$	1/8
4	$\max\{3 - 1/8(4 - 3 - 1 - 4), 0\} = 3\frac{1}{2}$	1/16
5	$\max\{3\frac{1}{2} - 1/16(4 - 0 - 1 - 4), 0\} = 3\frac{9}{16}$	1/32
6	$\max\{3\frac{9}{16} - 1/32(4 - 0 - 1 - 0), 0\} = 3\frac{15}{32} \approx 3.46875$	1/64
7	$\max\{3\frac{15}{32} - 1/64(4 - 0 - 1 - 4), 0\} = 3\frac{33}{64} \approx 3.516$	1/128
8	$\max\{3\frac{33}{64} - 1/128(4 - 0 - 1 - 0), 0\} = 3\frac{69}{128} \approx 3.539$	1/256
9	$\max\{3\frac{69}{128} - 1/256(4 - 0 - 1 - 0), 0\} = 3\frac{141}{256} \approx 3.551$	1/512

Note that u will continue increasing and the same solution will be optimal until u reach the value 4 or above. However the increase is at most $3(\frac{1}{512} + \frac{1}{1024} + \ldots) < 0.024$ and thus u will remain less than 4 and specifically does not approach $3\frac{1}{2}$. To see that $3(\frac{1}{512} + \frac{1}{1024} + \ldots) < 0.024$ note that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \to 1$. Subtracting $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{128} = 0.9921875$ we get $(\frac{1}{512} + \frac{1}{1024} + \ldots \to 0.0078125$, and the result follows.

Exercise 10.8

(i) Dualize budget-constraint

Strength: LP-relaxation.

Subproblem: Assignment problem - efficient even compared with LP-relaxation.

Lagrangian dual: Only one variable (relatively easy).

(ii) Dualize budget- and assignment-constraints

Strength: LP-relaxation.

Subproblem: Inspection, easier than (i).

Lagrangian dual: 1+|N|+|M| variables, harder than (i).

(iii) Dualize assignment-constraints

Strength: Potentially tighter than LP-relaxation.

Subproblem: One 0-1 knapsack problem NP-hard (easy NP-hard though).

Lagrangian dual: |N|+|M| variables, same as (ii).