The Min Cost Flow Problem

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Min Cost Flow - Terminology

- We consider a digraph $G = (V(G), E(G))$ in which each edge $e$ has a capacity $u_e \in \mathcal{R}_+$ and a unit transportation cost $c_e \in \mathcal{R}$.

- Each vertex $v$ furthermore has a demand $b_v \in \mathcal{R}$. If $b_v \geq 0$ then $v$ is called a sink, and if $b_v < 0$ then $v$ is called a source.

- We assume that $b(V) = \sum_{v \in V} b_v = 0$. 
Min Cost Flow - Example

Cost

Capacity

Capacity 5

2

10

3

8

Demand 5

Demand 10

Capacity 10

5

15

4

15

1

15
Min Cost Flow - Definition

The Min Cost Flow problem consists in supplying the sinks from the sources by a flow in the cheapest possible way:

\[
\min \sum_{e \in E} c_e x_e \\
f_x(v) = b_v \quad v \in V \\
0 \leq x_{vw} \leq u_{vw} \quad (u, w) \in E
\]

where \( f_x(v) = \sum_{(w, v) \in E} x_{wv} - \sum_{(w, v) \in E} x_{vw} \).
Special cases

Numerous flow problems can be stated as a Min Cost Flow problem:

1. The Transportation Problem
2. The Shortest Path Problem
3. The Max Flow Problem
Min Cost Flow - Dual LP

- The dual variables corresponding to the flow balance equations are denoted $y_v$, $v \in V$, and those corresponding to the capacity constraints are denoted $z_{vw}$, $(v, w) \in E$.

- The dual problem is now:

$$\begin{align*}
\max & \quad \sum_{v \in V} b_v y_v - \sum_{(v,w) \in E} u_{vw} z_{vw} \\
& \quad -y_v + y_w - z_{vw} \leq c_{vw} \iff (v, w) \in E \\
& \quad -c_{vw} - y_v + y_w \leq z_{vw} \iff (v, w) \in E \\
& \quad z_{vw} \geq 0 \quad (v, w) \in E
\end{align*}$$
\( \bar{c}_{vw} = c_{vw} + y_v - y_w \) is called the reduced cost for the edge \((v, w)\), and hence \(-c_{vw} - y_v + y_w \leq z_{vw} \) is equivalent to

\[-\bar{c}_{vw} \leq z_{vw} \]

When is the set of feasible solutions \(x, y\) and \(z\) optimal?
Optimality Conditions I

- If $u_e = +\infty$ (i.e. no capacity constraints for $e$) then $z_e$ must be 0 and hence $\bar{c}_e \geq 0$ just has to hold (primal optimality condition for the LP).

- If $u_e \neq +\infty$ then $z_e \geq 0$ and $z_e \geq -\bar{c}_e$ must hold. $z$ has negative coefficient in the objective function – hence the best choice for $z$ is as small as possible: $z_e = \max\{0, -\bar{c}_e\}$. Therefore, the optimal value of $z_e$ is uniquely determined from the other variables, and $z_e$ is “unnecessary” in the dual problem.
Optimality Conditions I - Comp. Slackness

- The complementary slackness conditions (each primal variable times the corresponding dual slack must equal 0, and each dual variable times the corresponding primal slack must equal 0 in optimum) now give:

\[ x_{vw} > 0 \Rightarrow -\bar{c}_{vw} = z_{vw} = \max(0, -\bar{c}_{vw}) \]

i.e. \((x_e > 0 \Rightarrow -\bar{c}_e \geq 0) \equiv (\bar{c}_e > 0 \Rightarrow x_e = 0)\)

and

\[ z_e > 0 \Rightarrow x_e = u_e \]

i.e. \((-\bar{c}_e > 0 \Rightarrow x_e = u_e) \equiv (\bar{c}_e < 0 \Rightarrow x_e = u_e)\)
Optimality Conditions II

Summing up: A primal feasible flow satisfying demands in sinks from sources respecting the capacity constraints is optimal if and only if we can find a dual solution $y_e, e \in E$ such that for all $e \in E$ it holds that:

$$
\bar{c}_e < 0 \Rightarrow x_e = u_e (\neq \infty)
$$

$$
\bar{c}_e > 0 \Rightarrow x_e = 0
$$

All pairs $(x, y)$ of optimal solutions satisfy these conditions — and so what?
Optimality Conditions III

For a legal flow $x$ in $G$, the residual graph is (like for Max Flow) a graph, in which the paths indicate **how flow excess can be moved** in $G$ given that the flow $x$ already is present. The only difference is that each edge has a cost assigned.
Optimality Conditions - Residual Graph

- The residual graph $G_x$ for $G$ wrt. $x$ is defined by

\[
V(G_x) = V(G) \\
E(G_x) = E_x = \\
\{(v, w) : (v, w) \in E \wedge x_{vw} < u_{vw}\} \cup \{(v, w) : (w, v) \in E \wedge x_{wv} > 0\}
\]
The unit cost $c'_{vw}$ for an edge with $x_{vw} < u_{vw}$ is $c_{vw}$, while $c'_{vw}$ for an edge with $x_{wv} > 0$ is $-c_{vw}$.

Note that a dicircuit with negative cost in $G_x$ corresponds to a negative cost circuit in $G$, if cost are added for forward edges and subtracted for backward edges.

Note that if a set of potentials $y_v$, $v \in V$ are given, and the cost of a circuit wrt. the reduced costs for the edges $(\overline{c}_{vw} = c_{vw} + y_v - y_w)$ are calculated, the cost remains the same as the original costs – the potentials are “telescoped” to 0.
Min Cost Flow - Negative cost circuits

- A primal feasible flow satisfying sink demands from sources and respecting the capacity constraints is optimal if and only if an $x$-augmenting circuit with negative $c$-cost (or negative $\bar{c}$-cost – there is no difference) does not exist.

- The idea behind the identification of optimal solutions in the network simplex algorithm.
Min Cost Flow - Network Simplex Algorithm

\[(G, c, u, x) \quad (G_x, c')x\]