The Min Cost Flow Problem

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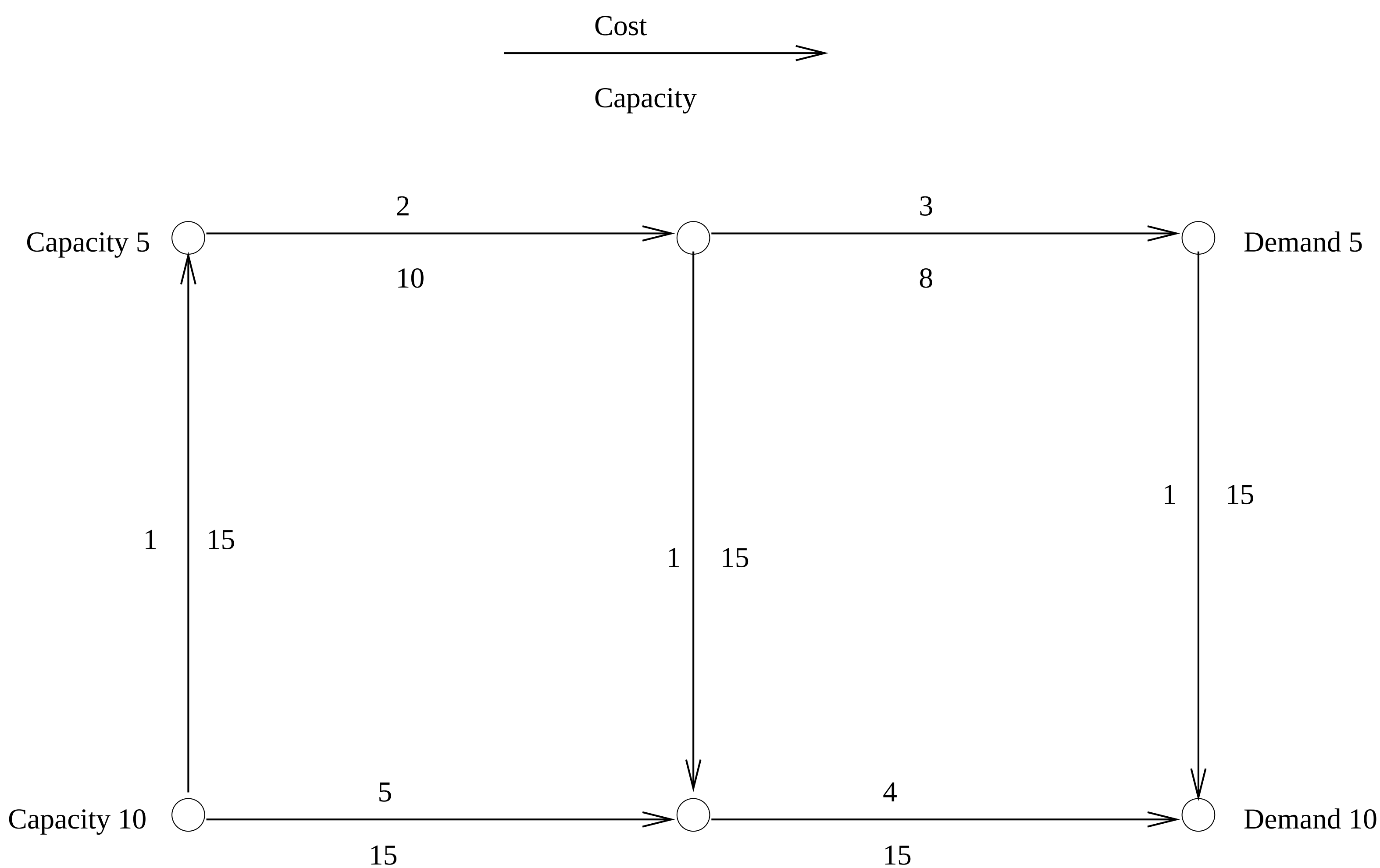
Min Cost Flow - Terminology

- We consider a digraph $G = (V(G), E(G))$ in which each edge e has a capacity $u_e \in \mathcal{R}_+$ and a unit transportation cost $c_e \in \mathcal{R}$.
- Each vertex v furthermore has a demand $b_v \in \mathcal{R}$. If $b_v \geq 0$ then v is called a **sink**, and if $b_v < 0$ then v is called a **source**.
- We assume that $b(V) = \sum_{v \in V} b_v = 0$.

Min Cost Flow - Example



Min Cost Flow - Definition

The Min Cost Flow problem consists in supplying the sinks from the sources by a flow in the cheapest possible way:

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ & f_x(v) = b_v & v \in V \\ & 0 \leq x_{vw} \leq u_{vw} & (u,w) \in E \end{array}$$

where $f_x(v) = \sum_{(w,v) \in E} x_{wv} - \sum_{(w,v) \in E} x_{vw}$.

Special cases

Numerous flow problems can be stated as a Min Cost Flow problem:

1. The Transportation Problem
2. The Shortest Path Problem
3. The Max Flow Problem

Min Cost Flow - Dual LP

- The dual variables corresponding to the flow balance equations are denoted $y_v, v \in V$, and those corresponding to the capacity constraints are denoted $z_{vw}, (v,w) \in E$.
- The dual problem is now:

$$\begin{aligned} \max \quad & \sum_{v \in V} b_v y_v - \sum_{(v,w) \in E} u_{vw} z_{vw} & \\ & -y_v + y_w - z_{vw} \leq c_{vw} \Leftrightarrow & (v,w) \in E \\ & -c_{vw} - y_v + y_w \leq z_{vw} & (v,w) \in E \\ & z_{vw} \geq 0 & (v,w) \in E \end{aligned}$$

$\bar{c}_{vw} = c_{vw} + y_v - y_w$ is called **the reduced cost** for the edge (v, w), and hence $-c_{vw} - y_v + y_w \leq z_{vw}$ is equivalent to

$$-\bar{c}_{vw} \leq z_{vw}$$

When is the set of feasible solutions x, y and z optimal?

Optimality Conditions I

- If $u_e = +\infty$ (i.e. no capacity constraints for e) then z_e must be 0 and hence $\bar{c}_e \geq 0$ just has to hold (primal optimality condition for the LP).
- If $u_e \neq +\infty$ then $z_e \geq 0$ and $z_e \geq -\bar{c}_e$ must hold. z has **negative** coefficient in the objective function – hence the best choice for z is as small as possible: $z_e = \max\{0, -\bar{c}_e\}$. Therefore, the optimal value of z_e is uniquely determined from the other variables, and z_e is "unnecessary" in the dual problem.

Optimality Conditions I - Comp. Slackness

- The complementary slackness conditions (each primal variable times the corresponding dual slack must equal 0, and each dual variable times the corresponding primal slack must equal 0 in optimum) now give:

$$x_{vw} > 0 \Rightarrow -\bar{c}_{vw} = z_{vw} = \max(0, -\bar{c}_{vw})$$

i.e. $(x_e > 0 \Rightarrow -\bar{c}_e \geq 0) \equiv (\bar{c}_e > 0 \Rightarrow x_e = 0)$

and

$$z_e > 0 \Rightarrow x_e = u_e$$

i.e. $(-\bar{c}_e > 0 \Rightarrow x_e = u_e) \equiv (\bar{c}_e < 0 \Rightarrow x_e = u_e)$

Optimality Conditions II

Summing up: A primal feasible flow satisfying demands in sinks from sources respecting the capacity constraints **is optimal if and only if** we can find a dual solution $y_e, e \in E$ such that for all $e \in E$ it holds that:

$$\bar{c}_e < 0 \Rightarrow x_e = u_e (\neq \infty)$$
$$\bar{c}_e > 0 \Rightarrow x_e = 0$$

All pairs (x, y) of optimal solutions satisfy these conditions — and so what?

Optimality Conditions III

- For a legal flow x in G, the residual graph is (like for Max Flow) a graph, in which the paths indicate **how flow excess can be moved** in G given that the flow x already is present. The only difference is that each edge has a cost assigned.

Optimality Conditions - Residual Graph

- The residual graph G_x for G wrt. x is defined by

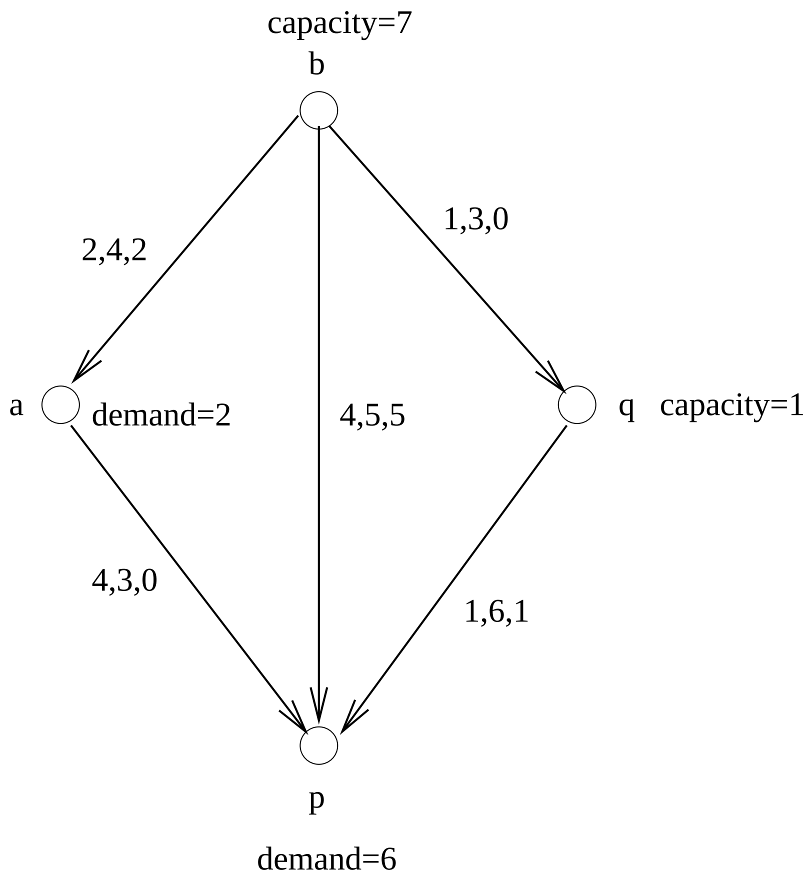
$$V(G_x) = V(G)$$
$$E(G_x) = E_x =$$
$$\{(v,w) : (v,w) \in E \wedge x_{vw} < u_{vw}\} \cup$$
$$\{(v,w) : (w,v) \in E \wedge x_{wv} > 0\}$$

- The unit cost c'_{vw} for an edge with $x_{vw} < u_{vw}$ is c_{vw}, while c'_{vw} for an edge with $x_{wv} > 0$ is $-c_{vw}$.
- Note that a dicircuit with negative cost in G_x corresponds to a negative cost circuit in G, if cost are added for forward edges and subtracted for backward edges.
- Note that if a set of potentials y_v, $v \in V$ are given, and the cost of a circuit wrt. the reduced costs for the edges $(\bar{c}_{vw} = c_{vw} + y_v - y_w)$ are calculated, the cost remains the same as the original costs – the potentials are "telescoped" to 0.

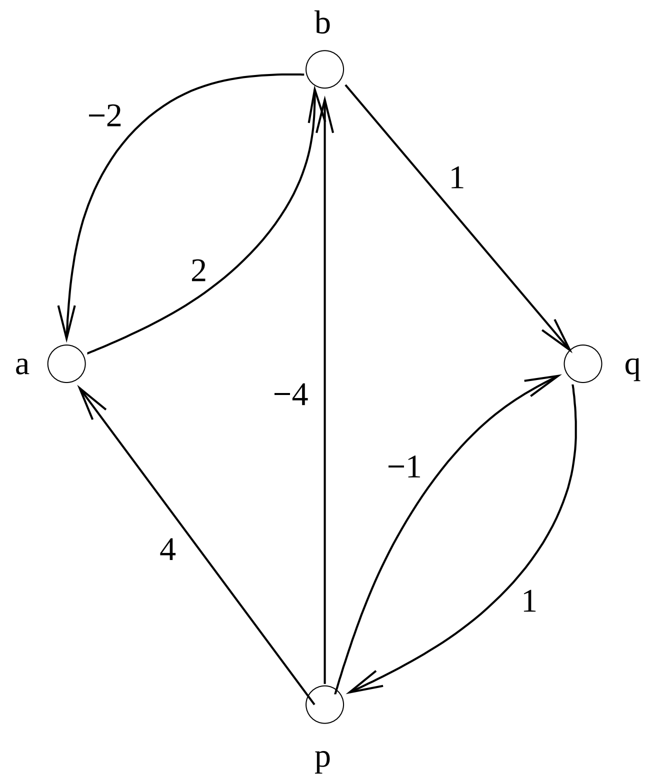
Min Cost Flow - Negative cost circuits

- A primal feasible flow satisfying sink demands from sources and respecting the capacity constraints **is optimal if and only if** an x-augmenting circuit with negative c-cost (or negative $\bar{c}$-cost – there is no difference) does not exist.
- The is the idea behind the identification of optimal solutions in the **network simplex algorithm**.

Min Cost Flow - Network Simplex Algorithm



(G,c,u,x)



(G_x, c')