Edge Colourings of Graphs

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For the chromatic index \( \chi'(G) \) of a (multi)graph \( G \) there are two natural lower bounds. On the one hand, \( \chi'(G) \geq \Delta(G) \) where \( \Delta(G) \) is the maximum degree of \( G \). On the other hand, \( \chi'(G) \geq W(G) \) where

\[
W(G) = \max_{H \subseteq G} \left[ \frac{|E(H)|}{\frac{1}{2} |V(H)|} \right].
\]

A graph \( G \) is called elementary if \( \chi'(G) = W(G) \). Goldberg conjectured around 1970 that every graph \( G \) is elementary provided that \( \chi(G) \geq \Delta(G) + 2 \). For an integer \( m \geq 3 \), let \( J_m \) denote the class of all graphs \( G \) such that

\[
\chi'(G) > \frac{m}{m-1} \Delta(G) + \frac{m-3}{m-1}.
\]

Shannon’s theorem implies that \( J_3 \) is empty. Furthermore, for every integer \( m \geq 3 \), we have \( J_m \subseteq J_{m+1} \) and the class \( J = \bigcup_{m=3}^{\infty} J_m \) consists of all graphs \( G \) such that \( \chi'(G) \geq \Delta(G) + 2 \).

A graph \( G \) is called critical if \( \chi'(H) < \chi'(G) \) for every proper subgraph \( H \) of \( G \). Jakobsen conjectured around 1975 that every critical graph in \( J_m \) has at most \( m - 2 \) vertices provided that \( m \geq 3 \) is odd. Up to now this conjecture is known to be true only for \( m \in \{5, 7, 9, 11\} \). In all these cases the proof of the statement that every graph in \( J_m \) has at most \( m - 2 \) vertices is based on a proof of the seemingly more general statement that every graph in \( J_m \) is elementary. This was proved, independently, by Sørensen for \( m = 5, 7 \) (unpublished), by Andersen for \( m = 5, 7 \) in 1977, by Goldberg for \( m = 5 \) in 1973 and for \( m = 9 \) in 1984, by Nishizeki and Kashiwagi for \( m = 11 \) in 1990, and, by Tashkinov for \( m = 11 \) in 2001. We use an extension of Tashkinov’s method to prove that every graph in \( J_{13} \) is elementary.