The Accommodating Ratio:

A performance measure for on-line algorithms

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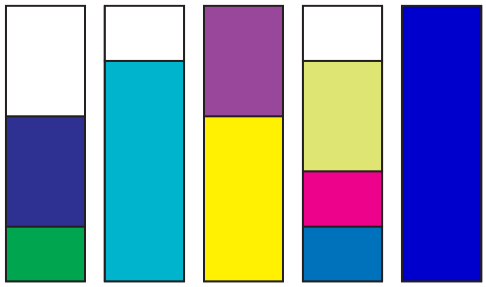
Outline

- Unit Price Bin Packing
- Worst-Fit has a better competitive ratio than First-Fit
- The Accommodating Ratio
- First-Fit has a better accommodating ratio than Worst-Fit
- Conclusion?

Unit Price Bin Packing

- n – the number of bins.
- k – the height of each bin.
- I – a sequence of integer-sized objects of size at most k



- Goal: Maximize the number of objects accepted.
- Off-line: (Coffman, Leung, Ting 1978).
- Applications: processor and storage allocation (Coffman, Leung 1979).

Algorithms

- First-Fit – Lowest numbered bin.
- Worst-Fit – Least full bin.
- OPT – optimal off-line algorithm

An algorithm is fair, if it never rejects a request, which could fit in some bin.

Easy observation: Fair algorithms always accept the first n requests.

The competitive ratio

- $\mathbb{A}(I)$ – how much an algorithm $\mathbb{A}$ earns on request sequence I.
- OPT(I) – how much OPT earns on I.

For a maximization problem, an on-line algorithm, $\mathbb{A}$, is c-competitive if there exists a constant b, such that $\mathbb{A}(I) \geq c \cdot \text{OPT}(I) - b$ for all I.

The competitive ratio

$$CR = \sup\{c \mid \mathbb{A} \text{ is } c\text{-competitive}\}$$

Results – Competitive ratio

Theorem 1 Worst-Fit's competitive ratio $\geq \frac{3}{2+k}$.

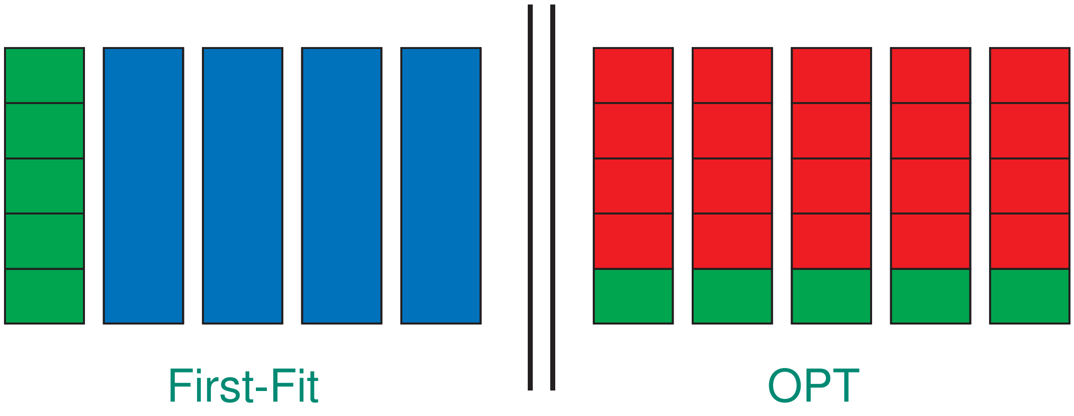
Theorem 2 First-Fit's competitive ratio $\leq \frac{2}{k}$.

Theorem 3 First-Fit's competitive ratio $\leq \frac{2}{k}$.

Proof Assume that k divides n.

Request sequence:

- n objects of size 1 Green
- $n - \frac{n}{k}$ objects of size k Blue
- $n(k-1)$ objects of size 1 Red



$$\begin{aligned} \frac{\text{ON-LINE}}{\text{OFF-LINE}} &\leq \frac{n + (n - \frac{n}{k})}{n + n(k-1)} \\ &\leq \frac{2n}{nk} \end{aligned}$$

$\square$

Theorem 4 Worst-Fit's competitive ratio $\geq \frac{3}{2+k}$.

Proof intuition

Observation: If for a sequence I, Worst-Fit leaves some bin less than full, $\frac{WF(I)}{OPT(I)} \geq \frac{3}{2+k}$.

Assume Worst-Fit has filled all bins.

X – objects accepted by Worst-Fit and OPT

Y – objects accepted by Worst-Fit, but not OPT

Z – objects accepted by OPT, but not Worst-Fit

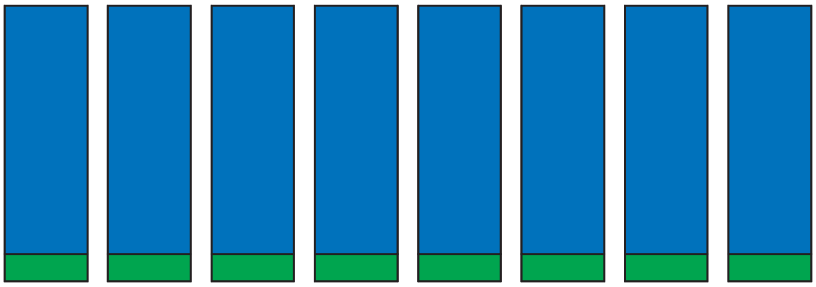
Z^f – objects in Z which come before the last object in Y

Claim: $|X| \geq 2(|Y| \setminus |Z^f|)$.

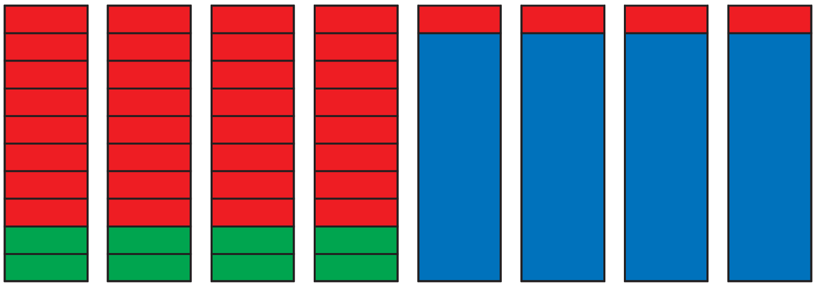
Example

Request sequence:

- n of size 1 Green
- n of size $k-1$ Blue
- $\frac{n}{2}(k-1)$ of size 1 Red



Accepted by WF



Accepted by OPT

Simplify: $|X| \geq 2|Y|$

Note: $|Z| \leq |Y| \cdot k$

Thus, the competitive ratio is

$$\frac{|X| + |Y|}{|X| + |Z|} \geq \frac{3|Y|}{2|Y| + |Y| \cdot k} = \frac{3}{2 + k}$$

Question: Do these sequences seem artificial?

What about request sequences where the objects all fit in the bins?

The Accommodating Ratio – AR

For a maximization problem, an on-line algorithm, $\mathbb{A}$, is c-accommodating if there exists a constant b, such that $\mathbb{A}(I) \geq c \cdot \text{OPT}(I) - b$ for all I which could be packed in n bins.

The accommodating ratio

$$AR = \sup\{c \mid \mathbb{A} \text{ is } c\text{-accommodating}\}$$

Compared with the previously defined competitive ratio:

For a maximization problem, an on-line algorithm, $\mathbb{A}$, is c-competitive if there exists a constant b, such that $\mathbb{A}(I) \geq c \cdot \text{OPT}(I) - b$ for all I.

The competitive ratio

$$CR = \sup\{c \mid \mathbb{A} \text{ is } c\text{-competitive}\}$$

Theorem 5 Any fair algorithm has $AR \geq \frac{1}{2}$.

Proof Consider any alg. $\mathbb{A}$ and any sequence I, which could be accommodated.

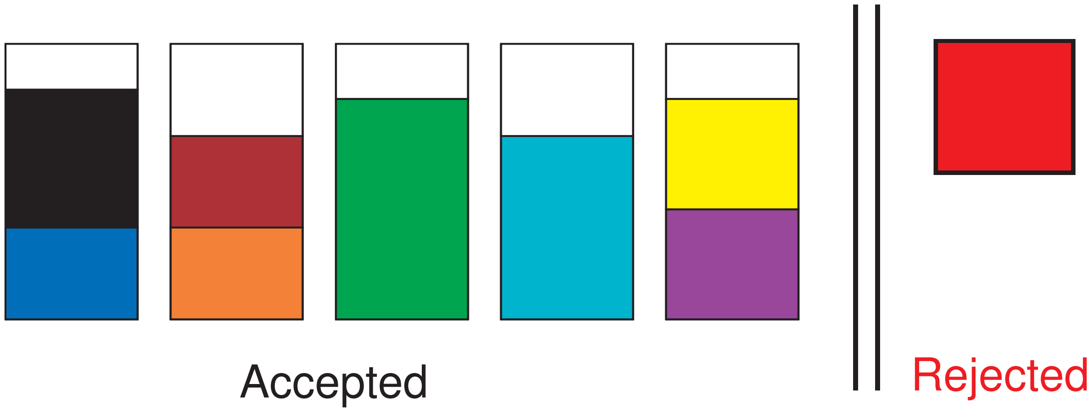
$A(\mathbb{A}, I) \stackrel{\text{def}}{\equiv}$ objects accepted by $\mathbb{A}$

$R(\mathbb{A}, I) \stackrel{\text{def}}{\equiv}$ objects rejected by $\mathbb{A}$

$$\frac{\text{ON-LINE}}{\text{OFF-LINE}} = \frac{|A(\mathbb{A}, I)|}{|A(\mathbb{A}, I)| + |R(\mathbb{A}, I)|}$$

Fact 1: WLOG $|A(\mathbb{A}, I)| \geq n$

Fact 2: $|R(\mathbb{A}, I)| < n$



$$\Rightarrow AR \geq \frac{n}{n + |R(\mathbb{A}, I)|} > \frac{n}{n+n} = \frac{1}{2}$$

$\square$

Results – Accommodating ratio

Theorem 6 Worst-Fit's accommodating ratio $\leq \frac{1}{2-\frac{1}{k}}$.

Theorem 7 First-Fit's accommodating ratio $\geq \frac{5}{8}$.

For comparison, recall that:

Theorem 8 Worst-Fit's competitive ratio $\geq \frac{3}{2+k}$.

Theorem 9 First-Fit's competitive ratio $\leq \frac{2}{k}$.

Theorem 10 Worst-Fit's accommodating ratio $\leq \frac{1}{2-\frac{1}{k}}$.

Proof Assume that k divides n.

Request sequence:

- n of size 1.
- $n - \frac{n}{k}$ of size k.

$$
\begin{aligned}
AR &\leq \frac{n}{n + n - \frac{n}{k}} \\
&= \frac{1}{2 - \frac{1}{k}}
\end{aligned}
$$

$\square$

Upper bound for First-Fit

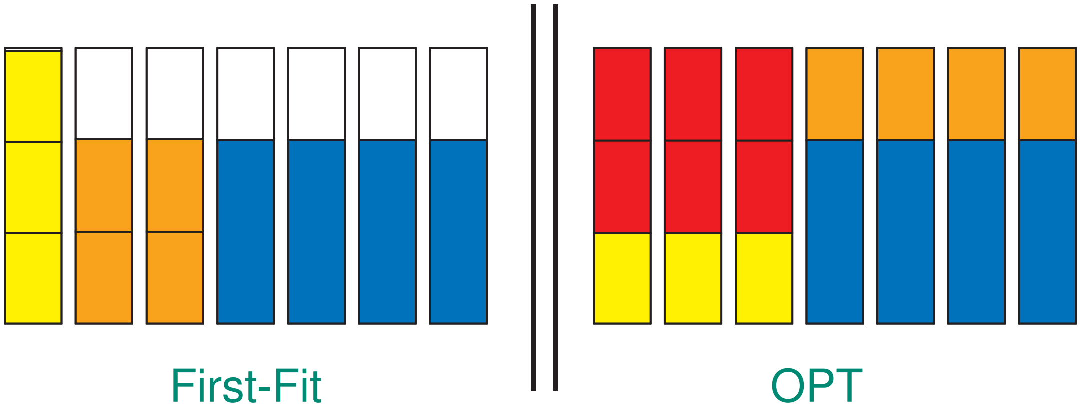
Theorem 11 First-Fit's accommodating ratio $\leq \frac{11}{17}$.

Proof Assume $3|k$, $k \geq 48$, $7|n$.

Request sequence:

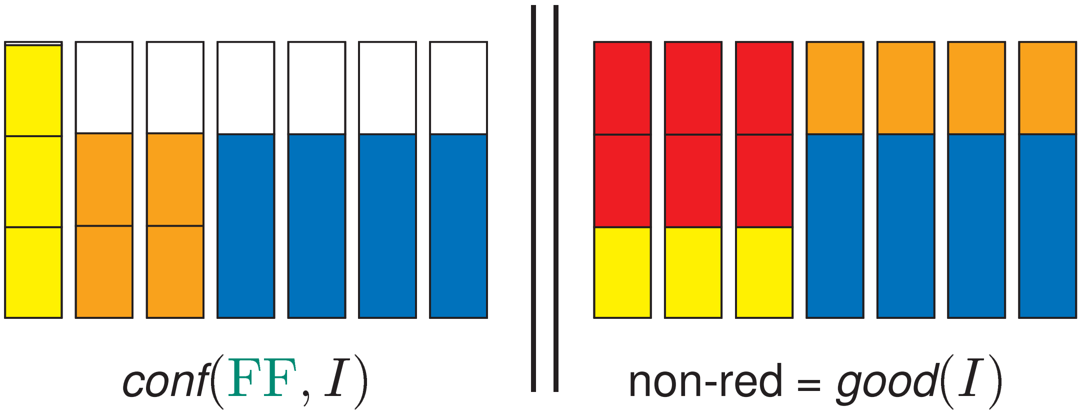
- $3 \times [\frac{k}{3} - 4]$ Yellow
- $4 \times [\frac{k}{3} + 1]$ Orange
- $4 \times [\frac{2k}{3} - 1]$ Blue
- $6 \times [\frac{k}{3} + 2]$ Red

$\square$



Lower bound First-Fit

Theorem 12 First-Fit's accommodating ratio $\geq \frac{3}{5}$.



Lemma 1 Suppose that all requests in I could be accommodated. The minimal number of objects which have to be moved from one bin to another, in order to change from *conf*(FF, I) to *good*(I), is an upper bound on the number of objects from I which First-Fit rejects.

proof

- Sequence $I =< s_1, s_2, \ldots, s_n >$.
- s — size of the smallest object in $R = R(\mathrm{FF}, I)$.

  Write $s_i = q_i s + r_i$, where $0 \le r_i \le s - 1$.
- e_j — Empty space in bin j (Before moves).

$$|R| \le \sum_{j=1}^{n} \lfloor \frac{1}{s}(e_j + \mathrm{vol}(out_j) - \mathrm{vol}(in_j)) \rfloor$$

$$= \sum_{j=1}^{n} \mathrm{net}_j$$

- net_j — net gain in bin j (The q_i's give no net gain.)

$$\begin{aligned} \mathrm{net}_j &\le \lfloor \frac{1}{s}(e_j + \sum_{s_i \in \mathrm{out}_j} r_i) \rfloor \\ &\le \lfloor \frac{1}{s}(s - 1 + |\mathrm{out}_j|(s-1)) \rfloor \\ &\le |\mathrm{out}_j| \end{aligned}$$

$$\begin{aligned} |R| &\le \sum_{j=1}^{n} \mathrm{net}_j \le \sum_{j=1}^{n} |\mathrm{out}_j| \\ &\le \text{Total number of moves.} \qquad \square \end{aligned}$$

Proof of Theorem 12 $(AR(FF) \geq \frac{3}{5})$

- B = set of objects alone in a bin by FF.
- $A = A(\text{FF}, I)$
- $|A| \geq 2n - |B|$
- $|A| \geq |B| + |R|$
- Combined: $|A| \geq n + \frac{|R|}{2}$

$$
\begin{aligned}
\frac{\textsc{On-line}}{\textsc{Off-line}} &= \frac{|A|}{|A| + |R|} \geq \frac{n + \frac{|R|}{2}}{n + \frac{3|R|}{2}} \\
&\geq \frac{2n + |R|}{2n + 3|R|} \geq \frac{2n + n}{2n + 3n}
\end{aligned}
$$

$\square$

Conclusion?

- Accommodating Ratio:

  First-Fit is better than Worst-Fit

- Competitive Ratio:

  Worst-Fit is better than First-Fit

Which algorithm is better?