Skriftlig Eksamen
Diskret Matematik (DM11)

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Alle sædvanlige hjælpemidler (lærebøger, notater etc.) samt brug af lommeregner er tilladt.
Eksamenssættet består af 6 opgaver på 7 nummererede sider (1–7). Bagerst findes en dansk oversættelse ligeledes på 7 nummerede sider (8-14).
Fuld besvarelse er besvarelse af alle 6 opgaver. De enkelte opgavers vægt ved bedømmelsen er angivet i point. Med mindre andet eksplicit er angivet, må man gerne referere til resultater fra lærebogen. Dette gælder også de opgaver, der har været stillet på ugesedlerne. Specielt må man, med mindre andet eksplicit er angivet, gerne begrunde en påstand med at henvise til, at det umiddelbart følger fra et resultat i lærebogen, eller én af de opgaver, der har været stillet på ugesedlerne (hvis dette altså er sandt !). Henvisninger til andre bøger (ud over lærebogen) om diskret matematik accepteres ikke som besvarelse af et spørgsmål!

Husk at begrunde alle dine påstande!

Problem No. 1 (20 points)

a) Prove that if $m^x \equiv m^y \pmod{n}$ for some positive integers x, y, m, and n, then $m^{x+k} \equiv m^{y+k} \pmod{n}$ for all integers $k \geq 0$.

b) Use Fermat's Little Theorem to compute the following two values:

$$5^{74} \pmod{7}$$
$$5^{74} \pmod{13}$$

c) Use your results from part (b) and the Chinese Remainder Theorem to find $5^{74} \pmod{91}$. (If you did not solve part (b) guess the result, choosing two different values, and solve part (c) using those values.)

Problem No. 2 (20 points)

In this problem we consider the set V defined as all triples of the form $(1, a_1, a_2)$, where $a_1, a_2 \in \{0, 1\}$.

a) Below we define three operations $\circ_1, \circ_2, \circ_3$ on the set V. For which of these operations $\circ_i, i \in \{1, 2, 3\}$ will $(V, \circ_i)$ be a group? Prove in each case which of the laws necessary for being a group are satisfied and which are violated.

   In all three definitions below, the addition $+$ is to be understood as **addition modulo 2**, i.e. $1 + 1 = 0$.

   i) $(1, a_1, a_2) \circ_1 (1, b_1, b_2) := (1, a_1 + b_2, a_2 + b_1)$;

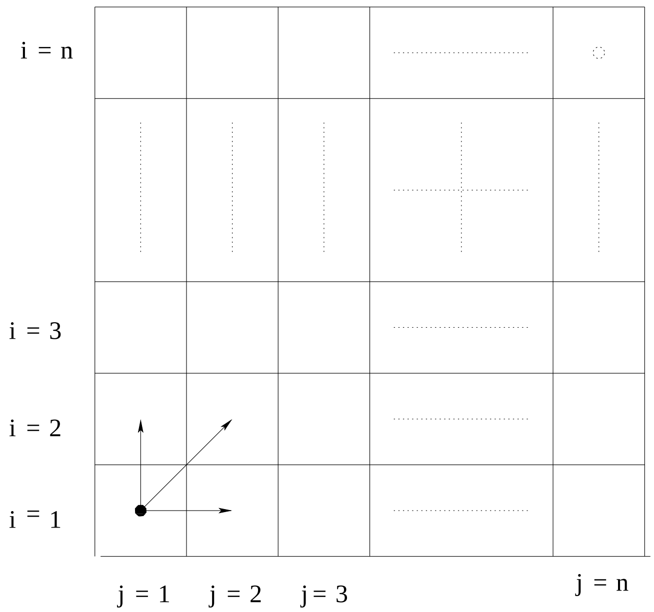
   ii) $(1, a_1, a_2) \circ_2 (1, b_1, b_2) := (1, a_1 + b_1 + 1, a_2 + b_2 + 1)$;

   iii) $(1, a_1, a_2) \circ_3 (1, b_1, b_2) := (1, a_1 + b_1, a_2 + b_2 + a_1 \cdot b_1)$.

b) For those $i \in \{1, 2, 3\}$ in part a) where $(V, \circ_i)$ turned out to be a group find an element x of maximal order and compute all cosets of the group $(V, \circ_i)$ with respect to the subgroup generated by x.

Problem No. 3 (20 points)

Let $n > 1$ be a natural number. We consider a chess-board with n rows and n columns. We number the rows from bottom to top and the columns from left to right, see the figure.



A token is placed on the field with coordinates $(1, 1)$ (left bottom corner) and should be moved to the field (n, n) (right top corner). There are three kinds of moves allowed:

i) move the token one row top-wards, i.e. move the token from position (i, j) to $(i + 1, j)$ for $i < 8$;

ii) move the token one column to the right, i.e. move the token from position (i, j) to $(i, j + 1)$ for $j < 8$;

iii) move the token one diagonal step, i.e. move the token from position (i, j) to $(i + 1, j + 1)$ for $i < 8, j < 8$.

a) For $n = 4$ determine the number of different paths along which we can move (under the above rules) the token from position $(1, 1)$ to $(4, 4)$?

b) For a general $n \in \mathbb{N}, n > 1$, what are the minimal and what the maximal necessary number of steps needed to move the token from $(1, 1)$ to (n, n)?

c) Suppose we want to use precisely k diagonal steps when moving from $(1, 1)$ to (n, n) (where, of course, k is related to the values you found in part b)). How many different possibilities are there? Give your result as a term depending on n and k.

d) Using parts b) and c) show that the number of different paths moving from $(1, 1)$ to (n, n) is given by

$$\sum_{j=0}^{n-1} \binom{n-1+j}{n-1-j} \cdot \binom{2 \cdot j}{j} .$$

Problem No. 4 (15 points)

Consider the finite set $A := \{a, b, c, d, e, f, g, h\}$. How many permutations of all elements of A exist in which none of the patterns ab, cd, ef, gh occurs?
Use the principle of inclusion and exclusion to calculate the result.

Problem No. 5 (15 points)

a) Suppose three events A_1, A_2, A_3 to be independent and each of it to occur with probability $\frac{1}{4}$. Compute the probabilities of the following events:

   i) at least one event among A_1 or A_2 occurs;

   ii) A_1 or A_2 occurs, but not both;

   iii) at least one of A_1, A_2 or A_3 occurs.

b) A hat contains n coins, s of which are fair and t of which are biased. The biased coins land heads with probability $\frac{2}{3}$.

   Suppose a coin is drawn at random from the hat and tossed twice. The results are (HEAD,TAIL). Under this assumption what is the probability that the coin was a fair one?

   Give the result in terms of s and t.

c) Suppose in part b) that we draw 120 times a coin from the hat, toss it, write down the result and then replace the coin into the hat.

   How do we have to choose s (in dependence of n), if the expected number of tails we get in the above experiment should be at least 50?

Problem No. 6 (10 points)

Let $f : \mathbb{N} \to \mathbb{R}$ be given by

$$f(n) = 5 \cdot 3^n + 6 \cdot 2^n$$

for all $n \in \mathbb{N}$ (i.e. $n \geq 1$).

a) Compute the initial values $f(1)$ and $f(2)$.

b) The function f satisfies a recurrence relation of the form:

$$f(n) = c_1 \cdot f(n-1) + c_2 \cdot f(n-2)$$

Compute c_1 and c_2.

c) Use induction to prove that f satisfies the recurrence relation defined using the constants you computed.