DM551

Algorithms and Probability

Joan Boyar

September 3, 2018

Format

- Lectures (in English)
  - Joan Boyar
  - Joan's office hours:
    Mondays 9:00–9:45, Thursdays 9:00–9:45
  - Questions in English or Danish
- Discussion sections
  - Kristine Vitting Klinkby Knudsen

Course exam

- Oral exam in January
- 2 exam assignments
  - 1st: you may work in groups of up to 3
  - 2nd: you must do it alone

Assignments

- no working with others not in your group
  (talk with me or Kristine)
- no late assignments (count as not done)
- turn in via Blackboard – 1 PDF file
- grading – count towards final grade with January exam

Assignments

- Begin early
- Ask if you do not understand
- Short, clear answers, but explain
- Do not reinvent the wheel –
  it is fine to make minor modifications to something from the textbook, just give a reference
- There are notes on how to write proofs on the course homepage

Discussion sections

- Read notes/textbook sections
  When asked to prepare before coming, prepare
- Think about problems
- Prepare at least one problem to present

Course Topics:

- Algorithm design and analysis
  - Randomized algorithms
  - Probabilistic analysis of algorithms
  - Linear recurrence relations
  - Universal hashing
  - String matching
- Combinatorics and probability
  - Counting techniques
  - Discrete probability

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?
  Answer: $(26 \cdot 26) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 26^2 \cdot 10^5$

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?
  Answer: $(26 \cdot 26) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 26^2 \cdot 10^5$
- Product rule: How many elements are there in $A_1 \times A_2 \times ... \times A_n$?

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?
  Answer: $(26 \cdot 26) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 26^2 \cdot 10^5$
- Product rule: How many elements are there in $A_1 \times A_2 \times ... \times A_n$?
  Answer: $|A_1| \cdot |A_2| \cdots |A_n|$.

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?
  Answer: $(26 \cdot 26) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 26^2 \cdot 10^5$
- Product rule: How many elements are there in $A_1 \times A_2 \times ... \times A_n$?
  Answer: $|A_1| \cdot |A_2| \cdots |A_n|$.
- How many bits strings are of length n?

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?
  Answer: $(26 \cdot 26) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 26^2 \cdot 10^5$
- Product rule: How many elements are there in $A_1 \times A_2 \times ... \times A_n$?
  Answer: $|A_1| \cdot |A_2| \cdots |A_n|$.
- How many bits strings are of length n?
  Answer: 2 possibilities for each bit, so 2^n.

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?
  Answer: $(26 \cdot 26) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 26^2 \cdot 10^5$
- Product rule: How many elements are there in $A_1 \times A_2 \times ... \times A_n$?
  Answer: $|A_1| \cdot |A_2| \cdots |A_n|$.
- How many bits strings are of length n?
  Answer: 2 possibilities for each bit, so 2^n.
- How many subsets are there of a set of size n?

Simple counting

- How many license plates can exist with 2 letters followed by 5 digits?
  Answer: $(26 \cdot 26) \cdot (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 26^2 \cdot 10^5$
- Product rule: How many elements are there in $A_1 \times A_2 \times \ldots \times A_n$?
  Answer: $|A_1| \cdot |A_2| \cdots |A_n|$.
- How many bits strings are of length n?
  Answer: 2 possibilities for each bit, so 2^n.
- How many subsets are there of a set of size n?
  Answer: There is a 1-1 correspondence between these subsets and the bit strings of length n, so 2^n.

Simple counting

- How many students got at least a grade 7 if 19 got 7, 10 got 10, and 4 got 12?

Simple counting

- How many students got at least a grade 7 if 19 got 7, 10 got 10, and 4 got 12?
  Answer: 33.

Simple counting

- How many students got at least a grade 7 if 19 got 7, 10 got 10, and 4 got 12?
  Answer: 33.
- Sum rule: How many elements are there in $A_1 \cup A_2 \cup \ldots \cup A_n$ if $A_i \cap A_j = \emptyset$ for $i \neq j$?

Simple counting

- How many students got at least a grade 7 if 19 got 7, 10 got 10, and 4 got 12?
  Answer: 33.
- Sum rule: How many elements are there in $A_1 \cup A_2 \cup ... \cup A_n$ if $A_i \cap A_j = \emptyset$ for $i \neq j$?
  Answer: $|A_1| + |A_2| + \ldots + |A_n|$.

Simple counting

- How many students got at least a grade 7 if 19 got 7, 10 got 10, and 4 got 12?
  Answer: 33.
- Sum rule: How many elements are there in $A_1 \cup A_2 \cup \ldots \cup A_n$ if $A_i \cap A_j = \emptyset$ for $i \neq j$?
  Answer: $|A_1| + |A_2| + \ldots + |A_n|$.
- What fraction of the students got at least a grade of 7 if $\frac{19}{71}$ got 7, $\frac{10}{71}$ got 10 and $\frac{4}{71}$ got 12?

Simple counting

- How many students got at least a grade 7 if 19 got 7, 10 got 10, and 4 got 12?
  Answer: 33.
- Sum rule: How many elements are there in $A_1 \cup A_2 \cup ... \cup A_n$ if $A_i \cap A_j = \emptyset$ for $i \neq j$?
  Answer: $|A_1| + |A_2| + \ldots + |A_n|$.
- What fraction of the students got at least a grade of 7 if $\frac{19}{71}$ got 7, $\frac{10}{71}$ got 10 and $\frac{4}{71}$ got 12?
  Answer $\frac{33}{71}$.

Simple counting

- How many functions $f : A \to B$, where $|A| = m$, $|B| = n$?

Simple counting

- How many functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Answer: There are n possible values for $f(a)$ $\forall a \in A$, so n^m.

Simple counting

- How many functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Answer: There are n possible values for $f(a)$ $\forall a \in A$, so n^m.
- How many 1-1 functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Note: $m \leq n$.

Simple counting

- How many functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Answer: There are n possible values for $f(a)$ $\forall a \in A$, so n^m.
- How many 1-1 functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Note: $m \leq n$.
  Answer: n possibilities for the 1st
  $n - 1$ possibilities for the 2nd
  $n - 2$ possibilities for the 3rd
  ... $n - m + 1$ possibilities for last
  So $\prod_{i=0}^{m-1}(n - i) = \frac{n!}{(n-m)!}$.

Simple counting

- How many functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Answer: There are n possible values for $f(a)$ $\forall a \in A$, so n^m.
- How many 1-1 functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Note: $m \leq n$.
  Answer: n possibilities for the 1st
  $n - 1$ possibilities for the 2nd
  $n - 2$ possibilities for the 3rd
  ... $n - m + 1$ possibilities for last
  So $\prod_{i=0}^{m-1}(n - i) \ = \ \frac{n!}{(n-m)!}$.
- The Inclusion-Exclusion Principle. (2 sets)
  $|A \cup B| \ = \ |A| + |B| - |A \cap B|$

Simple counting

- How many functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Answer: There are n possible values for $f(a)\ \forall a \in A$, so n^m.
- How many 1-1 functions $f : A \to B$, where $|A| = m$, $|B| = n$?
  Note: $m \leq n$.
  Answer: n possibilities for the 1st
  $n - 1$ possibilities for the 2nd
  $n - 2$ possibilities for the 3rd
  ... $n - m + 1$ possibilities for last
  So $\prod_{i=0}^{m-1}(n - i) = \frac{n!}{(n-m)!}$.
- The Inclusion-Exclusion Principle. (2 sets)
  $|A \cup B| = |A| + |B| - |A \cap B|$
- Tree diagrams.

The Pigeonhole Principle

Thm. [The Pigeonhole Principle] If $\geq k+1$ pigeons go into k holes, then ≥ 1 hole has ≥ 2 pigeons.

The Pigeonhole Principle

Thm. [The Pigeonhole Principle] If $\geq k+1$ pigeons go into k holes, then ≥ 1 hole has ≥ 2 pigeons.

Thm. [The Generalized Pigeonhole Principle] If N pigeons go into k holes, then ≥ 1 hole has $\geq \lceil N/k \rceil$ pigeons.

The Pigeonhole Principle

Thm. [The Pigeonhole Principle] If $\geq k+1$ pigeons go into k holes, then ≥ 1 hole has ≥ 2 pigeons.

Thm. [The Generalized Pigeonhole Principle] If N pigeons go into k holes, then ≥ 1 hole has $\geq \lceil N/k \rceil$ pigeons.

Pf. Suppose no hole contains $> \lceil N/k \rceil - 1$ pigeons. Then, the number of pigeons is

$$\leq k(\lceil N/k \rceil - 1) \; < \; k((N/k+1)-1) \; = \; N.$$

Contradiction. So ≥ 1 hole has $\geq \lceil N/k \rceil$ pigeons. $\square$

Examples:

- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
  Suppose $B \subset A$, $|B| = 5$. Must there be 2 integers in B which sum to 9?

Examples:

- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
  Suppose $B \subset A$, $|B| = 5$. Must there be 2 integers in B which sum to 9?
  Yes. Consider $\{\{1, 8\},\ \{2, 7\},\ \{3, 6\},\ \{4, 5\}\}$. Since there are 5 elements from these 4 subsets, some subset must have 2.

Examples:

- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
  Suppose $B \subset A$, $|B| = 5$. Must there be 2 integers in B which sum to 9?
  Yes. Consider $\{\{1, 8\},\ \{2, 7\},\ \{3, 6\},\ \{4, 5\}\}$. Since there are 5 elements from these 4 subsets, some subset must have 2.
- Suppose you have 10 red balls and 10 blue balls. How many do I have to take before I am sure I have ≥ 3 of the same color?

Examples:

- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
  Suppose $B \subset A$, $|B| = 5$. Must there be 2 integers in B which sum to 9?
  Yes. Consider $\{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\}$. Since there are 5 elements from these 4 subsets, some subset must have 2.
- Suppose you have 10 red balls and 10 blue balls. How many do I have to take before I am sure I have ≥ 3 of the same color?
  5. There are 2 sets. 5 is the least N s.t. $\lceil N/2 \rceil = 3$.

Examples:

- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
  Suppose $B \subset A$, $|B| = 5$. Must there be 2 integers in B which sum to 9?
  Yes. Consider $\{\{1, 8\},\ \{2, 7\},\ \{3, 6\},\ \{4, 5\}\}$. Since there are 5 elements from these 4 subsets, some subset must have 2.
- Suppose you have 10 red balls and 10 blue balls. How many do I have to take before I am sure I have ≥ 3 of the same color?
  5. There are 2 sets. 5 is the least N s.t. $\lceil N/2 \rceil = 3$.
- $3, 19, 13, 7, 12, 8, 4, 9, 29, 31, 2, 16, 1, 15, 21, 5, 6$. Does this sequence have a subsequence of length 5 which is strictly increasing or strictly decreasing?

Examples:

- Let $A = \{1,2,3,4,5,6,7,8\}$.
  Suppose $B \subset A$, $|B| = 5$. Must there be 2 integers in B which sum to 9?
  Yes. Consider $\{\{1,8\},\ \{2,7\},\ \{3,6\},\ \{4,5\}\}$. Since there are 5 elements from these 4 subsets, some subset must have 2.
- Suppose you have 10 red balls and 10 blue balls. How many do I have to take before I am sure I have ≥ 3 of the same color?
  5. There are 2 sets. 5 is the least N s.t. $\lceil N/2 \rceil = 3$.
- $3, 19, 13, 7, 12, 8, 4, 9, 29, 31, 2, 16, 1, 15, 21, 5, 6$. Does this sequence have a subsequence of length 5 which is strictly increasing or strictly decreasing?
  Yes. $3, 19, 13, 7, 12, 8, 4, 9, 29, 31, 2, 16, 1, 15, 21, 5, 6$.
  3,7,8,29,31. With $n^2 + 1$ elements, must have $n + 1$ increasing or decreasing. ($n = 4$, $n + 1 = 5$, $n^2 + 1 = 17$.)

Ramsey Theory

Suppose there are 6 people.

Each pair is either friends or enemies.

There is a subset of 3 — all friends or all enemies.

Ramsey Theory

Suppose there are 6 people.

Each pair is either friends or enemies.

There is a subset of 3 — all friends or all enemies.

Why? Consider one person P. There are 5 other people. By the Generalized Pigeonhole Principle, either at least 3 are friends or at least 3 are enemies of P.

Suppose A, B, and C are friends of P. If any pair are friends, we have a subset of 3 friends. If they are all enemies, they form a subset of 3 enemies.

Suppose A, B, and C are enemies of P. If any pair are enemies, we have a subset of 3 enemies. If they are all friends, they form a subset of 3 friends.

Ramsey Theory

Suppose there are 6 people.

Each pair is either friends or enemies.

There is a subset of 3 — all friends or all enemies.

Why? Consider one person P. There are 5 other people. By the Generalized Pigeonhole Principle, either at least 3 are friends or at least 3 are enemies of P.

Suppose A, B, and C are friends of P. If any pair are friends, we have a subset of 3 friends. If they are all enemies, they form a subset of 3 enemies.

Suppose A, B, and C are enemies of P. If any pair are enemies, we have a subset of 3 enemies. If they are all friends, they form a subset of 3 friends.

$R(m, n)$ number needed to ensure at least m friends or n enemies.