Permutations

Permutation – an ordering of some objects.

r-**permutation** – an ordering of r objects from a set.

Example: $5, 2, 3$ — 3-permutation
of $\{1, 2, 3, 4, 5\}$.

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Thm. The number of r-permutations of n elements is

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = n!/(n-r)!$$

Pf. Same argument as for the number of 1-1 functions $f : A \to B$, where $|A| = r$, $|B| = n$, since A can be thought of as the position number. $\square$

Combinations

r-**combination** – an unordered subset of r objects from a set.

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Thm. The number of r-combinations of n elements is

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Pf. Any r-permutation is an ordering of an r-combination. There are $P(r, r) = r!$ ways to order an r-permutation. Thus,

$$P(n, r) = C(n, r) \cdot P(r, r)$$

and

$$C(n, r) = \frac{n!}{(n-r)!} / r! = \frac{n!}{r!(n-r)!} \qquad \square$$

Examples

Example 1: A *traveling salesman* starts in city 1, has to visit 5 other cities (6 in all), each exactly once, and then return home. How many tours are there?

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Example 2: Suppose there are 36 numbers in a lottery and you have to guess 7 correct to win. How many different possibilities are you choosing from?

Answer: $\frac{36!}{7!29!} = 8,347,680$. So you have less than 1 chance in 8,000,000 of winning big.

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Notation: $\binom{n}{r} = C(n,r)$ is a *binomial coefficient*.

Binomial Coefficients

Thm. [Pascal's Identity] $n \geq k \geq 1$.

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

Pf.

$$\begin{aligned}
C(n+1,k) &= \tfrac{(n+1)!}{k!(n+1-k)!} \\
&= \tfrac{(n+1)n!}{k!(n-k)!(n+1-k)} \\
&= \tfrac{(n+1-k)n!}{k!(n-k)!(n+1-k)} \\
&\qquad + \tfrac{k \cdot n!}{k!(n-k)!(n+1-k)} \\
&= C(n,k) + \tfrac{n!}{(k-1)!(n+1-k)!} \\
&= C(n,k) + C(n,k-1) \qquad \square
\end{aligned}$$

Identities for Binomial Coefficients

Thm. $n \geq 1$. $\sum_{k=0}^{n} C(n,k) = 2^n$.

Pf. Consider a set with n elements.

There are $C(n,k)$ different subsets of size k.

Thus, the total number of subsets is $\sum_{k=0}^{n} C(n,k)$.

The number of subsets of a set of size n is 2^n. $\square$

Identities for Binomial Coefficients

Thm. [Vandermonde's Identity] $n \geq r \geq 0$. $m \geq r$.

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k).$$

Pf. Suppose $|A| = n$, $|B| = m$.

Let $C = A \cup B$. $|C| = m + n$.

$C(m+n, r) =$ number of ways to choose r elements from C.

Same as choosing k from A and $r-k$ from B.

$C(m, r-k)C(n, k)$ ways — for fixed k.

$\sum_{k=0}^{r} C(m, r-k)C(n, k)$ ways in all.

Thus, $C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k)$. $\square$

Identities for Binomial Coefficients

Thm. $n \geq 0$.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Pf. Recall Vandermonde's Identity:

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k).$$

Let $m = r = n$.

$$C(n+n, n) = \sum_{k=0}^{n} C(n, n-k)C(n, k) = \sum_{k=0}^{n} \binom{n}{k}^2. \qquad \square$$

Identities for Binomial Coefficients

Thm. $n \geq r \geq 0$.

$$\binom{n+1}{r+1} = \sum_{j=0}^{n} \binom{j}{r}.$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7} \\
 & & & & & & & \cdots & & & & & & &
\end{array}
$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\cdots
\end{array}
$$

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & \cdots & & & & & & &
\end{array}
$$

$$
\begin{aligned}
(x+y)^1 &= x + y \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{aligned}
$$

The Binomial Theorem

Thm. [The Binomial Theorem] $n \geq 1$. x, y variables.

$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j$

$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$

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$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$

Alternate proof:

$(x+y)^n \ = \ (x+y)(x+y)\cdots(x+y)$ n terms.

Result — choose an x or a y from each term.

How many ways to get $x^{n-j} y^j$?

Choose which terms give a y — $C(n,j)$.

Thus, $(x+y)^n = \sum_{j=0}^{n} C(n,j) x^{n-j} y^j$ for $n \geq 1$. $\square$

Using the binomial theorem

Thm. $n \geq 1$. $\sum_{k=0}^{n} C(n,k) = 2^n$.

Pf. $2^n = (1+1)^n = \sum_{k=0}^{n} C(n,k) 1^{n-k} 1^k = \sum_{k=0}^{n} C(n,k)$ $\square$

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Pf. $2^n = (1+1)^n = \sum_{k=0}^{n} C(n,k) 1^{n-k} 1^k = \sum_{k=0}^{n} C(n,k)$ $\square$

Thm. $n \geq 1$. $\sum_{k=0}^{n} (-1)^k C(n,k) = 0$.

Pf. $0 = (1+(-1))^n = \sum_{k=0}^{n} C(n,k) 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k C(n,k)$ $\square$

Combinations with repetition

Example

Experiment: choose balls from: 3 red. 4 green. 5 blue.

How many ways to get 1 red ball and 2 green balls? $3 \cdot 4^2$

Note: not interested in order.

Represent result as: $n_1 \mid n_2 \mid n_3$, where

n_1 — number of red balls

n_2 — number of green balls

n_3 — number of blue balls

Combinations with repetition

Example

We wanted: $1 \mid 2 \mid 0$.

Could also represent as $*|**|$.

How many distinct results are there if 3 balls chosen? $(n_1 \mid n_2 \mid n_3)$

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How many distinct results are there if 3 balls chosen? $(n_1 \mid n_2 \mid n_3)$

The number of ways to write down 3 *s and 2 $|$s.

(For example, $*|**|$ or $|***|$.)

There are 5 places. Choose 3 places for stars.

Answer: $\binom{5}{3} = \frac{5!}{3!2!} = 10$.

Combinations with repetition

Example

Same experiment with n types of balls and r choices.

(An r-combination from n elements, with repetition.)

There are r stars and $n-1$ bars.

Number of distinct results is $\binom{n+r-1}{r}$.

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Same experiment with n types of balls and r choices.

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Example

How many ways can you choose k integers

≥ 0, $(n_1, n_2, \ldots, n_k)$, such that $\displaystyle\sum_{i=1}^{k} n_i \;=\; n$?

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Example

How many ways can you choose k integers

≥ 0, $(n_1, n_2, \ldots, n_k)$, such that $\displaystyle\sum_{i=1}^{k} n_i \;=\; n$?

Answer: $\binom{k+n-1}{n}$.

Combinations with repetition

Example

How many nonnegative solutions are there to

$$n_1 + n_2 + n_3 + n_4 = 7?$$

Answer: $\binom{4+7-1}{7} = \frac{10!}{7!3!} = 120.$

Combinations with repetition

A bag with 3 red balls, 4 green balls, and 5 blue balls.

Consider all red balls the same, etc.

How many orderings are there of the balls?

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There are (3+4+5)! orderings.

Given one ordering, $(x_1, x_2, ..., x_{12})$,

there are 3!4!5! which are identical to it.

The number of distinct orderings is $\frac{12!}{3!4!5!} = 27,720$.

Combinations with repetition

Example

How many ways to put n pigeons in k holes,

if the ith hole should have n_i pigeons?

Answer: $\binom{n}{n_1}$ for the first hole.
$\binom{n-n_1}{n_2}$ for the second hole.
$\binom{n-n_1-n_2}{n_3}$ for the third hole.

Etc. Multiply these together to get:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$