Conditional Probability

Thm. [Baye's Theorem] For events A, B, with $p(A) > 0$, $p(B) > 0$, $p(A|B) = \frac{p(A)p(B|A)}{p(B)}$.

Pf. By the def of conditional probability,

$p(A \cap B) = p(B)p(A|B)$.

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Divide both sides by $p(B)$. $\square$

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Example: With 2 fair dice, what is the probability that the sum is 7, given that both dice are ≥ 3?

Answer: $p(\text{sum is } 7 \mid \text{both} \geq 3) = \frac{p(\text{sum is } 7) \cdot p(\text{both } {\geq}3 \mid \text{sum is } 7)}{p(\text{both } {\geq}3)} = \frac{\frac{1}{6} \cdot \frac{1}{3}}{\frac{2}{3} \cdot \frac{2}{3}} = 1/8$.

Expectations

Recall:

Def. A *random variable* is a function $f : S \to \mathbb{R}$.

Def. For a finite sample space $S = \{s_1, s_2, ..., s_n\}$, the *expected value* of the random variable $X(s)$ is

$$E(X) = \sum_{i=1}^{n} p(s_i)X(s_i).$$

Def. For a countably infinite sample space $S = \{s_1 \mid i \geq 1\}$, the *expected value* of the random variable $X(s)$ is $E(X) = \sum_{i=1}^{\infty} p(s_i)X(s_i)$.

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Example: What is the expected number of successes in n Bernoulli trials? Probability of success $= p$. Probability of failure $=$
$q = 1 - p$.

Expectations

Answer:

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{n} k \cdot p(X = k) \\
&= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \left(\frac{n!}{k!(n-k)!} \right) p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \left(\frac{n(n-1)!}{k(k-1)!(n-1-k+1)!} \right) p^k q^{n-k} \\
&= n \sum_{k=1}^{n} \left(\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} \right) p^k q^{n-k} \\
&= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{n-k}
\end{aligned}
$$

Expectations

$$\begin{aligned}
E[X] &= n\sum_{k=1}^{n}\binom{n-1}{k-1}p^k q^{n-k} \\
&= np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1} q^{n-k} \\
&= np\sum_{j=0}^{n-1}\binom{n-1}{j}p^{j} q^{n-1-j} \\
&= np(p+q)^{n-1} \\
&= np. \qquad \Box
\end{aligned}$$

Expectations

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Answer:

$$\begin{aligned}
\sum_{i=1}^{\infty} i q^{i-1} p &= \sum_{i=1}^{\infty} i q^{i-1} - \sum_{i=1}^{\infty} i q^{i} \\
&= \sum_{j=0}^{\infty} (j+1) q^{j} - \sum_{i=1}^{\infty} i q^{i} && (j = i-1) \\
&= 1 + \sum_{j=1}^{\infty} (j+1-j) q^{j} && . \\
&= 1 + \sum_{j=1}^{\infty} q^{j} \; = \; 1 + \left(\sum_{j=0}^{\infty} q^{j}\right) - 1 \\
&= \frac{1}{1-q} \\
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Answer:

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&= \sum_{j=0}^{\infty}(j+1)q^{j} - \sum_{i=1}^{\infty} iq^{i} \qquad\qquad (j = i-1) \\
&= 1 + \sum_{j=1}^{\infty}(j+1-j)q^{j} \qquad\qquad . \\
&= 1 + \sum_{j=1}^{\infty} q^{j} \;=\; 1 + (\sum_{j=0}^{\infty} q^{j}) - 1 \\
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\end{aligned}
$$

With a fair die, the expected number of throws before a 1 is 6.