Chebyshev's Inequaltiy

Thm. [Chebyshev's Inequality] Let X be a random variable on sample space S, with probability function p, and $r > 0$. Then

$$p(|X(s) - E[X]| \geq r) \leq V[X]/r^2$$

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Deviation from the mean when counting tails

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Suppose $r = \sqrt{n}$.

$$p(|X(s) - E[X]| \geq r) \leq V[X]/r^2$$

$$p(|X(s) - n/2| \geq \sqrt{n}) \leq (n/4)/n = 1/4$$

Results are not always useful, but this worked OK.

The Principle of Inclusion-Exclusion

$A_1, A_2, \ldots, A_n$ finite sets.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2|$$
$$-|A_1 \cap A_2| - |A_1 \cap A_2| + |A_1 \cap A_2 \cap A_3|.$$

Thm.[The Principle of Inclusion-Exclusion]

$$|A_1 \cup A_2 \cup \ldots \cup A_n|$$
$$= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$
$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$
$$- \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|.$$

Pf. (by induction on n)

Basis step: $n = 2$. Proven earlier.

Inductive step:

Applications

Counting number of elements with or without the properties $P_1, P_2, ..., P_n$:

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Counting number of elements with or without the properties $P_1, P_2, ..., P_n$:

$N(P_{i_1} P_{i_2} ... P_{i_k})$ – number of elements with properties $P_{i_1}, P_{i_2}, ..., P_{i_k}$

$N(P'_{i_1} P'_{i_2} ... P'_{i_k})$ – number of elements with none of the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$.

A_i — subset of elements with property P_i.

N — total number of elements

$$N(P_{i_1} P_{i_2} ... P_{i_k}) = |A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}|$$

$$N(P'_{i_1} P'_{i_2} ... P'_{i_k}) = N - |A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_k}|$$

$$N(P'_{i_1} P'_{i_2} ... P'_{i_k}) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) - \ldots + (-1)^n N(P_1, P_2, ..., P_n).$$

How many solutions are there to $n_1 + n_2 + n_3 + n_4 = 7$, if you must have $n_1, n_2, n_3 \leq 3$ and $n_4 \leq 4$.

Property P_1 is $n_1 > 3$.

Property P_2 is $n_2 > 3$.

Property P_3 is $n_3 > 3$.

Property P_4 is $n_4 > 4$.

$$
\begin{aligned}
N(P_1'P_2'P_3'P_4') = N \\
&- N(P_1) - N(P_2) - N(P_3) - N(P_4) \\
&+ N(P_1P_2) + N(P_1P_3) + N(P_1P_4) \\
&+ N(P_2P_3) + N(P_2P_4) + N(P_3P_4) \\
&- N(P_1P_2P_3) - N(P_1P_2P_4) \\
&- N(P_1P_3P_4) - N(P_2P_3P_4) \\
&+ N(P_1P_2P_3P_4)
\end{aligned}
$$

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- $N(P_1 P_2)$ = number with $n_1 \geq 4$ and $n_2 \geq 4$. This is impossible, so 0.
- All intersections of at least 2 are impossible.

Thus, $N(P_1' P_2' P_3' P_4') = 120 - 3(20) - 10 + 0 = 50$.

The Sieve of Eratosthenes

To get a list of primes $\leq B$:

Create an empty list P.
Create a list L of the integers $2..B$.
while L is not empty **do**
    Remove p — smallest element in L.
    Insert p in P.
    Delete all multiples of p from L.

How many primes are there ≤ 19?

How many numbers are in P if $B = 19$?

Property $P_i(x)$ is: ith prime$< x \leq 19$
and ith prime divides x.

Every composite ≤ 19 is divisible by a prime $\leq \sqrt{19} \leq 5$.

The Sieve of Eratosthenes

Answer: $N(P_1'P_2'P_3') = N$
$$- N(P_1) - N(P_2) - N(P_3)$$
$$+ N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

- $N = 18$.
- $N(P_1) = 8$ (2 divides x)
- $N(P_2) = 5$ (3 divides x)
- $N(P_3) = 2$ (5 divides x)
- $N(P_1P_2) = 3$ (6 divides x)
- $N(P_1P_3) = 1$ (10 divides x)
- $N(P_2P_3) = 1$ (15 divides x)
- $N(P_1P_2P_3) = 0$ (30 divides x)

$N(P_1'P_2'P_3') = 18 - 8 - 5 - 2 + 3 + 1 + 1 - 0 = 8.$

(Same result if P_3 is not included.)

Derangements

Example: Peter likes betting. He hears:

- Team A is expected to win.
- Team B is expected to be 2nd.
- Team C is expected to be 3rd.
- Team D is expected to be 4th.
- Team E is expected to be 5th.

Peters bets on these 5 events.

In how many ways can he lose all bets?

A *derangement* is a permutation with no object in its original position.

Property P_i — place i was correct.

Want $D = N(P_1' P_2' P_3' P_4' P_5')$.

Derangements

$$\begin{aligned} D &= N - \sum_{i=1}^{5} N(P_i) + \sum_{1 \le i < j \le 5} N(P_iP_j) - \sum_{1 \le i < j < k \le 5} N(P_iP_jP_k) \\ &+ \sum_{1 \le i < j < k < l \le 5} N(P_iP_jP_kP_l) - N(P_1P_2P_3P_4P_5). \end{aligned}$$

$N = 5!.$

$N(P_i) = (5-1)! \quad \forall i.$

$N(P_iP_j) = (5-2)! \quad \forall i, j.$

$N(P_iP_jP_k) = (5-3)! \quad \forall i, j, k.$

$N(P_iP_jP_kP_l) = (5-4)! \quad \forall i, j, k, l.$

$N(P_1P_2P_3P_4P_5) = (5-5)!.$

How many terms are there in each sum?

Derangements

$$\sum_{i=1}^{5} N(P_i) \text{ — } \binom{5}{1}.$$

$$\sum_{1\le i<j\le 5} N(P_iP_j) \text{ — } \binom{5}{2}.$$

$$\sum_{1\le i<j<k\le 5} N(P_iP_jP_k) \text{ — } \binom{5}{3}.$$

$$\sum_{1\le i<j<k<l\le 5} N(P_iP_jP_kP_l) \text{ — } \binom{5}{4}.$$

$$N(P_1P_2P_3P_4P_5) \text{ — } \binom{5}{5}.$$

$$D = 5! - (5-1)!\tfrac{5!}{1!(5-1)!} + (5-2)!\tfrac{5!}{2!(5-2)!}$$
$$- (5-3)!\tfrac{5!}{3!(5-3)!} + (5-4)!\tfrac{5!}{4!(5-4)!} - (5-5)!\tfrac{5!}{5!(5-5)!}$$

$$D = 5!\left(1 - \tfrac{1}{1!} + \tfrac{1}{2!} - \tfrac{1}{3!} + \tfrac{1}{4!} - \tfrac{1}{5!}\right) = 44.$$

Thm. The number of derangements of a set with n elements is

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n\frac{1}{n!}\right)$$

Derangements

Suppose all permutations were equally likely.

What is the probability of a derangement?

$$D_n/n! = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!}$$

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The infinte sum

$$1 - \frac{1}{1!} + \frac{1}{2!} - ... + (-1)^n \frac{1}{n!} + ...$$

gives $1/e \approx 0.368$.

$\frac{1}{e} - \frac{1}{(n+1)!} \leq \frac{D_n}{n!} \leq \frac{1}{e} + \frac{1}{(n+1)!}$.

The probability of at least one object being fixed is approximately $1 - 1/e \approx 0.632$.