Institut for Matematik og Datalogi
Syddansk Universitet

Exam Assignment 1
Algorithms and Probability — 2020

This is the first of two sets of problems (assignments) which together with the oral exam in January constitute the exam in DM551. This first set of problems may be solved in groups of up to three. You are encouraged to form groups; if you have trouble finding people to work with and would like help, write to your TA (Sissel or Martin).

The assignment is due at 10:15 on Thursday, October 22. You may write this either in Danish or English. Write your full name (or names if you do it together — up to three people may work together) clearly on the first page of your assignment (on the top, if it's not a cover page). Turn it in as a PDF file via Blackboard through your DM551 course. The assignment hand-in is in the menu for the course and is called "SDU Assignment". Keep the receipt it gives you proving that you turned your assignment in on time. Blackboard will not allow you to turn in an assignment late.

Cheating on this assignment is viewed as cheating on an exam. Do not talk with anyone outside of your group (or Joan Boyar or your TA) about the assignment, and do not show your solutions to anyone outside your group. If you have questions about the assignment, come to Joan Boyar or your TA.

Note that material for the last three problems will be covered in class before the assignment is due, but you can start on the rest with no problem.

Assignment 1

Do the following problems. Write clear, complete answers, but not longer than necessary. Explain all answers, especially those where the answer is just a number. Do not reduce your answers to decimal numbers or calculate factorials of numbers that are not small. Leave the answer as a formula or fraction when relevant.

1. In this problem, you should show the following concerning giving change: Given a set of coin values $\{v_1, v_2, \ldots, v_n\}$ and a target value t, suppose

you have a sequence of coins (coin values, possibly including duplicates) $X = \langle x_1, x_2, \ldots, x_h \rangle$, containing the smallest possible number of coins, such that $t = \sum_{i=1}^{h} x_i$. Suppose $v_1 < v_2 < \cdots < v_n$. If $t > v_n^2$, then the coin v_n must be used at least once in X.

(An example instance of this problem is: coin values $\{1, 5, 10, 25, 100\}$, target value 132, solution sequence $\langle 1, 1, 5, 25, 100 \rangle$. Note that $t = 132 \leq 100^2 = v_5^2$, so we did not have the case that 100 is necessarily used at least once because of the result, but it was used once anyways.) Assume in what follows that $t > v_n^2$, and that $t = \sum_{i=1}^{h} x_i$, but do not assume that X is the sequence having this property containing the smallest number of coins.

(a) Show that $h > v_n$.

(b) Consider the values $p_k = \sum_{i=1}^{k} x_i$, $1 \leq k \leq h$. Show that the exist $1 \leq i < j \leq h$ such that $p_i \equiv p_j \pmod{v_n}$.

(c) Show that there is some subsequence of X, $\langle x_\ell, x_{\ell+1}, \ldots, x_{m-1}, x_m \rangle$ such that $\sum_{i'=\ell}^{m} x_{i'} \equiv 0 \pmod{v_n}$.

(d) Show that replacing $\langle x_\ell, x_{\ell+1}, \ldots, x_{m-1}, x_m \rangle$ in the original sequence X by some copies of the value v_n gives a different sequence, no longer than X, giving the target value t.

(e) Argue that the shortest sequence of coins contains at least one coin of value v_n.

2. We know from rules concerning exponentiation that $2^n \cdot 3^n = 6^n$. Show how to obtain this result using the technique in the proof of Corollary 1 to the binomial theorem (page 439 is Rosen's textbook), along with the result in that corollary. (Start with 6^n, find two appropriate numbers that add together to make 6, and apply the binomial theorem to that.)

3. A bakery has 4 types of cupcakes: chocolate with cream cheese frosting, chocolate with chocolate frosting, carrot with cream cheese frosting, and banana with chocolate frosting. How many ways are there to choose:

   (a) 5 cupcakes?

   (b) 8 cupcakes with at least one of each type of cupcake?

4. How many solutions does $x_1 + x_2 + x_3 = 9$ have, where x_1, x_2, and x_3 are nonnegative integers?

5. Generating permutations:

   (a) Show how you use Algorithm 1, in Section 6.6 of Rosen's textbook to find the next permutation of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, in ascending lexicographic order, after the permutation 12587643.

   (b) Explain which changes are needed to in Algorithm 1 to find the previous, instead of the next permutation.

6. Choose a random letter x from the string "permutation" and a random letter y from the string "combination" (the probability of choosing a "t" from "permutation" should be twice the probability of choosing an "a" from "permutation"). What is the probability that $x = y$? Explain your answer.

7. Suppose that three fair (6-sided) dice are rolled, one at a time. What are the following:

   (a) the probability that the sum of the three numbers is at least 14?

   (b) the expected number of times that three dice need to be rolled before the sum of the three numbers is at least 14?

   (c) the probability that the sum of the three numbers is at least 14, given that the first of the three dice has value 5? (Explain how you computed this using conditional probabilities.)

8. In this problem you should analyze a Monte Carlo algorithm to determine if two polynomials, $f(x)$ and $g(x)$, are such that $f(x) - g(x)$ is a constant function (the result of the subtraction is the same, regardless of x). Assume that the only access you have to the two polynomials is through two procedures F and G, which each have exactly one integer argument x, where F computes $f(x)$ and G computes $g(x)$. Thus, you could compute both f and g on the value 3 by running $F(3)$ and $G(3)$. If two polynomials both have degree at most k ($x^4 - 5x^3 - 2x + 9$ has degree 4, the highest exponent on x), then if $f(x) - g(x)$ gives the same result on more than k different values, $f(x) - g(x)$ is a constant function. Assume that you have a large upper bound n on the degrees of f and g, but the bound is so large that you cannot try $n+1$ different values. The algorithm is as follows:

procedure MonteCarloCheckPolynomials(F, G, n, M,m):
{ Input: F, G evaluate functions f and g of degree at most n;
random integers between 1 and M, inclusive, are chosen m times}
{ Output: "not constant" if $f(x) - g(x)$ is proven not constant;
"constant" otherwise" }
$c \leftarrow F(0) - G(0)$
$i \leftarrow 0$
repeat
let $x \in_R \{1, M\}$ { x chosen from uniform distribution of integers between 1 and M, inclusive }
if $F(x) - G(x) \neq c$, **then return** "not constant"; halt
$i \leftarrow i + 1$
until $(i = m)$
return "constant"

Compute the probably of error, assuming that the polynomials both have degree n (for both possible answers your algorithm can give).

9. Consider rolling a fair die.

   (a) What is the exact probability that either a 1 or a 3 is rolled?

   (b) Use Chebyshev's inequality to give an upper bound on the probability that you roll either a 1 or a 6.

   (c) Use Markov's inequality to give an upper bound on the probability that you roll a 1.

10. Find the number of positive integers not exceeding 139 that are not divisible by 3, 5, or 7.

11. Suppose that p and q are odd primes with $p \neq q$ and r is a positive integer. Use the principle of inclusion-exclusion to compute $\phi(pq^r)$, the number of positive integers less than pq^r which are not divisible by p or q (are relatively prime to pq^r).