Institut for Matematik og Datalogi
Syddansk Universitet

Exam Assignment 2
Algorithms and Probability — 2020

This is the second of two sets of problems (assignments) which together with the oral exam in January constitute the exam in DM551. You must do this assignment individually, not in groups.

The assignment is due at 8:00 on Friday, December 11. You may write this either in Danish or English. Write your full name clearly on the first page of your assignment (on the top, if it's not a cover page). Turn it in as a PDF file via Blackboard through your DM551 course. The assignment hand-in is in the menu for the course and is called "SDU Assignment". Keep the receipt it gives you proving that you turned your assignment in on time. Blackboard will not allow you to turn in an assignment late.

Cheating on this assignment is viewed as cheating on an exam. Do not talk with anyone either than the TAs, Martin or Sissel, or Joan about the assignment, and do not show your solutions to anyone else. If you have questions about the assignment, come to Joan, Martin, or Sissel.

Assignment 2

Do the following problems. Write clear, complete answers, but not longer than necessary. Explain all answers, especially those where the answer is very short.

1. Consider the following error in designing the `Select` algorithm given on page 728 of Kleinberg and Tardos: Rather choosing the splitter from the current set S, it chooses it uniformly randomly from the original set of n numbers, S, each time. It only splits the current S into (S^-, S^+), when the splitter is in the current S.

   (a) Write this modified algorithm in pseudocode, using notation similar to that of `Select` in Kleinberg and Tardos. (Think about how to determine if the splitter is in the current S or not.)

(b) Show that your algorithm is correct, in the sense that it always eventually returns the kth largest element in S.

(c) What is the expected running time of this modified algorithm? Explain why your upper bound holds.

2. (a) Give a counterexample to the following statement:

   For every flow network $G = (V, E)$ with source s and sink t, there is always an edge $e \in E$ such that increasing the capacity of e increases the value of the maximum flow.

   (b) Give an example of a flow network $G = (V, E)$ with source s and sink t, where there is an edge $e \in E$ such that increasing the capacity of e increases the value of the maximum flow.

   (c) Devise an $O(|E|)$ algorithm which, given a flow network $G = (V, E)$ with source s and sink t, along with a maximum flow in G, finds an edge e such that increasing the capacity of e increases the value of the maximum flow or else determines that no such edge exists. (If you cannot find an $O(|E|)$ algorithm, find a slower polynomial time algorithm.)

3. Let $G = (V, E)$ be a digraph. Let s and t be distinct vertices. A set P of paths from s to t is called *edge disjoint* if no two paths in P have an edge in common. A set Q of edges is called a *blocker of $s - t$ paths* if every path from s to t includes an edge in Q.

   Using the max-flow min-cut theorem, prove: $\max |P| = \min |Q|$, where P ranges over edge-disjoint sets of $s - t$ paths, and Q ranges over blockers of $s - t$ paths.

4. Let $a_n = 8a_{n-1} + 9a_{n-2}$ with $a_0 = 0$ and $a_1 = 10$. Solve for a_n and show that your result holds for a_2.

5. Let $a_n = 5a_{n-1} - 6a_{n-2} + 3n$ with $a_0 = 7$ and $a_1 = 11$. Solve for a_n and show that your result holds for a_2.

6. In this problem, you will approximate the number of binary strings of length n with at most three consecutive zeros.

   Let t_k be the number of strings with this property (the length is k, and there are at most three consecutive zeros) that begin and end with a 1.

(a) Find a linear recurrence for these t_k. (If you can't find this yourself, ask me for the result, but you will not get credit for this part.) Hint: $t_1 = 1$, $t_2 = 1$, $t_3 = 2$, $t_4 = 4$. Look at suffixes it could end with, 11, 101, 1001, 10001.

(b) Find the characteristic equation for the recurrence relation for t_k. (If you do not succeed in solving the previous question, choose a recurrence relation giving a characteristic equation of degree 4 and solve the remaining using that. Make sure that there is a unique largest real characteristic root.)

(c) Find an approximation to the unique largest real characteristic root. Give upper and lower bounds that differ by at most $\frac{1}{10{,}000}$. You may use Maple, or any program to do this. Explain how you did it.

(d) Assume that this unique largest real characteristic root, r, is such that the value r^n is asymptotically larger than the values r_i^n for any other characteristic root r_i of the characteristic equation. Use big-O and Ω notation to give an asymptotic upper and lower bounds on how large t_n is.

(e) Relate this result to the original problem of counting the number of binary strings of length n with at most three consecutive zeros. (Give upper and lower bounds using asymptotic notation, and explain why they hold.)