Institut for Matematik og Datalogi
Syddansk Universitet

October 14, 2020
JFB

DM551/MM851 – Algorithms and Probability – 2020
Lecture 8

Lecture, October 1

We finished section 7.4. We also covered sections 8.5 and and up through, but not including derangements in section 8.6.

Lecture, October 7, in U23

We will finish with derangements and cover section 13.1 in chapter 13 in *Algorithm Design* by Kleinberg and Tardos (KT, available in Course Materials in Blackboard).

Lecture, October 19, in U23

We will cover section 13.2 in chapter 13 in KT.

Discussion sessions canceled on October 21

Problems to be discussed on October 28

1. Finish any problems not finished on October 8/9.
2. Exercises on pages 782–793: 1, 3, 4. (For problem 3, for any given process P_i, what is the expected number of executions of this protocol before P_i enters the set S for the first time? Answer this for both parts, a. and b., of the question.)
3. Discuss Solved exercise 1 on page 776 of Kleinberg and Tardos. This is not available in Course Materials in Blackboard. The problem is as follows:

   We use an undirected graph $G = (V, E)$ to model the problem where there are $n = |V|$ small devices, each of which uses wireless communication to communicate with d of the other other nearby devices. The vertices in V are the devices, and there are edges between vertices representing pairs of devices within communication range of each other. Thus, each vertex has exactly d neighbors.

   Some of the devices should be given an *uplink transmitter* so they can send data to the main station, but we want to use as few uplink transmitters as possible to still get all of the data from all of the devices. Thus, in our graph, we want a subset

$S \subseteq V$ such that for all $v \in V$, either $v \in S$ or v has a neighbor u such that $u \in S$. Such a set S is call a *dominating set*. If the devices corresponding to vertices in S all get uplink transmitters, then all devices can either send data to the main station directly, or send data to a device that can send data directly.

Finding a minimum sized dominating set is an NP-hard problem, so one does not expect to find a polynomial time deterministic algorithm to solve it.

In this problem, consider a randomized algorithm which chooses a set S of $k = \frac{cn\ln(n)}{d+1}$ vertices from V uniformly at random. The constant c will be discussed later. You should show through the following that S is a dominating set with high probability.

In the following, assume that the vertices in S are chosen one at a time and it is possible to choose the same vertex more than once. A vertex v is said to dominate a vertex u if they are the same vertex or u is a neighbor of v. A set S' dominates u if some vertex in S' dominates u.

(a) Show that any dominating set in G has size at least $\frac{n}{d+1}$. This says something about how good our solution is. What?

(b) Let $D[v,t]$ be the event that the tth random vertex chosen dominates v. What is the probability of $D[v,t]$?

(c) Let D_v be the event that S dominates v. What is the probability of the complement of this event, $\overline{D_v}$?

(d) Show that $\left(\frac{1}{e}\right)^{c\ln(n)} = \frac{1}{n^c}$ is an upper bound on the probability of $\overline{D_v}$.

(e) Let A be the event that S is a dominating set. Use the Union Bound to give an upper bound on the probability of $\overline{A}$.

(f) Discuss what value c should have. Is there a threshold effect here?