DM551/DM851

Algorithms and Probability

Joan Boyar

September 1, 2020

Format

- Lectures (in English)
  - Joan Boyar
  - Joan's office hours:
    Mondays 9:00–9:45, Thursdays 9:00–9:45
  - Questions in English or Danish
- Discussion sections
  - H1: Martin Lorenzen
  - H2: Sissel Banke
- Blackboard information
  - Home Page - lecture notes, slides, assignments
  - Course Information - code for ebook discount
  - Course Materials - parts of textbook, other notes
  - Course Recordings - recordings
  - Discussion Board - students–students, students–Joan, students–TA
  - SDU Assignment - for submitting assignment solutions

Course format

- Prerequisites – Discrete Methods,
  Algorithms and Data Structures
- Online vs. offline
  - The first five weeks are online
    (see the "Online" comment in the course schedule) – Zoom
  - After that there should be some at SDU
  - You are muted when starting session
  - Use the chat for asking and answering questions
  - To make a comment when asked, unmute yourself
    Please also do this for my (or the TA's) technical problems.
  - Show up on time when you can, to ask questions and to help me or the TA.
  - In Socrative (for polls) the room number is 415439. Use b.socrative.com/login/student

Course exam

- Prerequisite for taking the exam
- Oral exam in January
- 2 exam assignments
  - 1st: you may work in groups of up to 3
  - 2nd: you must do it alone

Assignments

- no working with others not in your group
  (talk with me, Sissel, or Martin)
- no late assignments (count as not done)
- turn in via Blackboard – 1 PDF file
- grading – count towards final grade with January exam

Assignments

- Begin early
- Ask if you do not understand
- Short, clear answers, but explain
- Do not reinvent the wheel –
  it is fine to make minor modifications to something from the textbook, just give a reference
- There are notes on how to write proofs on the course homepage

Discussion sections

- Read notes/textbook sections
  When asked to prepare before coming, prepare.
  There is some preparation for September 9.
- Think about problems
- Prepare at least one problem to present

Poll

Will you be able to attend online events on time if they are preceded by an event at SDU ending 15 minutes earlier?

A. Yes (I live close to SDU or will do it at SDU)
B. No, but I will (usually) be at most 15 minutes late
C. No, but I will (usually) be at most 30 minutes late
D. No, but I will (usually) be at most 45 minutes late
E. No.

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Poll

If an event in our class occurs immediately before or after a SDU event in another course, which do you prefer?

A. Not changing it
B. Using a Doodle to find a new time that day (8:15 is OK)
C. Using a Doodle to find a new time another day

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Poll

Answer true (yes) or false (no) to the following:

Would you be happy with (accept) using the first half of a discussion section before a lecture and the last half of one after (say on September 9)?

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Poll

Answer true (yes) or false (no) to the following:

Would you be happy with (accept) switching discussion sections (those in h1 attending h2 and vice versa for a certain discussion section, say the first in week 39)?

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Course Topics:

- Algorithm design and analysis
  - Randomized algorithms
  - Probabilistic analysis of algorithms
  - Linear recurrence relations
  - Universal hashing
  - Network flows (applications, such as matching)
  - String matching
- Combinatorics and probability
  - Counting techniques
  - Discrete probability

Simple counting

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  100000001 - length 9 - first and last elements in set, others not

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  Answer $\frac{33}{71}$.

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  A. m^n
  B. n^{n-m}
  C. $\frac{n!}{(n-m)!}$
  D. $\frac{n!}{m!(n-m)!}$
  E. $\frac{n^m}{m!}$

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  Answer: n possibilities for the 1st
  $n - 1$ possibilities for the 2nd
  $n - 2$ possibilities for the 3rd
  ... $n - m + 1$ possibilities for last
  So $\prod_{i=0}^{m-1}(n - i) \; = \; \frac{n!}{(n-m)!}$.

Simple counting

The Inclusion-Exclusion Principle. (2 sets)

$$|A \cup B| \;=\; |A| + |B| - |A \cap B|$$

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Tree diagrams.

The Pigeonhole Principle

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Thm. [The Generalized Pigeonhole Principle] If N pigeons go into k holes, then ≥ 1 hole has $\geq \lceil N/k \rceil$ pigeons.

Pf. Suppose no hole contains $> \ \lceil N/k \rceil - 1$ pigeons. Then, the number of pigeons is

$$\leq k(\lceil N/k \rceil - 1) \ < \ k((N/k+1)-1) \ = \ N.$$

Contradiction. So ≥ 1 hole has $\geq \lceil N/k \rceil$ pigeons. $\square$

Examples:

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

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Let $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

Suppose $B \subset C$, $|B| = 8$.

Is the following statement true or false?

There must be 2 integers in B which sum to 15.

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- $3, 19, 13, 7, 12, 8, 4, 9, 29, 31, 2, 16, 1, 15, 21, 5, 6$. Does this sequence have a subsequence of length 5 which is strictly increasing or strictly decreasing?

Examples:

- Suppose you have 10 red balls and 10 blue balls. How many do I have to take before I am sure I have ≥ 3 of the same color?
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- $3, 19, 13, 7, 12, 8, 4, 9, 29, 31, 2, 16, 1, 15, 21, 5, 6$. Does this sequence have a subsequence of length 5 which is strictly increasing or strictly decreasing?
  Yes. $3, 19, 13, 7, 12, 8, 4, 9, 29, 31, 2, 16, 1, 15, 21, 5, 6$.
  3,7,8,29,31. With $n^2 + 1$ elements, must have $n + 1$ increasing or decreasing. $(n = 4,\ n + 1 = 5,\ n^2 + 1 = 17.)$

Ramsey Theory

Suppose there are 6 people.

Each pair is either friends or enemies.

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Why? Consider one person P. There are 5 other people. By the Generalized Pigeonhole Principle, either at least 3 are friends or at least 3 are enemies of P.

Suppose A, B, and C are friends of P. If any pair are friends, we have a subset of 3 friends. If they are all enemies, they form a subset of 3 enemies.

Suppose A, B, and C are enemies of P. If any pair are enemies, we have a subset of 3 enemies. If they are all friends, they form a subset of 3 friends.

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$R(m, n)$ number needed to ensure at least m friends or n enemies.