Algorithms and Probability

Lecture 2

September 3, 2020

Permutations

Permutation – an ordering of some objects.

r-**permutation** – an ordering of r objects from a set.

Example: $5, 2, 3$ — 3-permutation
of $\{1, 2, 3, 4, 5\}$.

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Thm. The number of r-permutations of n elements is

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = n!/(n-r)!$$

Pf. Same argument as for the number of 1-1 functions $f : A \to B$, where $|A| = r$, $|B| = n$, since A can be thought of as the position number. $\square$

Combinations

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Thm. The number of r-combinations of n elements is

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Pf. Any r-permutation is an ordering of an r-combination. There are $P(r, r) = r!$ ways to order an r-permutation. Thus,

$$P(n, r) = C(n, r) \cdot P(r, r)$$

and

$$C(n, r) = \frac{n!}{(n-r)!} / r! = \frac{n!}{r!(n-r)!} \qquad \square$$

Examples

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Notation: $\binom{n}{r} = C(n,r)$ is a *binomial coefficient*.

Binomial Coefficients

Thm. [Pascal's Identity] $n \geq k \geq 1$.

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

Pf.

$$\begin{aligned}
C(n+1,k) &= \tfrac{(n+1)!}{k!(n+1-k)!} \\
&= \tfrac{(n+1)n!}{k!(n-k)!(n+1-k)} \\
&= \tfrac{(n+1-k)n!}{k!(n-k)!(n+1-k)} \\
&\qquad + \quad \tfrac{k \cdot n!}{k!(n-k)!(n+1-k)} \\
&= C(n,k) + \tfrac{n!}{(k-1)!(n+1-k)!} \\
&= C(n,k) + C(n,k-1) \qquad \square
\end{aligned}$$

Identities for Binomial Coefficients

Thm. $n \geq 1$. $\sum_{k=0}^{n} C(n, k) = 2^n$.

Pf. Consider a set with n elements.

There are $C(n, k)$ different subsets of size k.

Thus, the total number of subsets is $\sum_{k=0}^{n} C(n, k)$.

The number of subsets of a set of size n is 2^n. $\square$

Identities for Binomial Coefficients

Thm. [Vandermonde's Identity] $n \geq r \geq 0$. $m \geq r$.

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k).$$

Pf. Suppose $|A| = n$, $|B| = m$.

Let $C = A \cup B$. $|C| = m + n$.

$C(m+n, r)$ = number of ways to choose r elements from C.

Same as choosing k from A and $r - k$ from B.

$C(m, r-k)C(n, k)$ ways — for fixed k.

$\sum_{k=0}^{r} C(m, r-k)C(n, k)$ ways in all.

Thus, $C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k)$. $\square$

Identities for Binomial Coefficients

Thm. $n \geq 0$.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Pf. Recall Vandermonde's Identity:

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k).$$

Let $m = r = n$.

$$C(n+n, n) = \sum_{k=0}^{n} C(n, n-k)C(n, k) = \sum_{k=0}^{n} \binom{n}{k}^2. \qquad \square$$

Identities for Binomial Coefficients

Thm. $n \geq r \geq 0$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
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$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\cdots$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

...

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & \cdots & & & & & & &
\end{array}
$$

$$
\begin{aligned}
(x+y)^1 &= x + y \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{aligned}
$$

The Binomial Theorem

Thm. [The Binomial Theorem] $n \geq 1$. x, y variables.

$(x+y)^n = \sum_{j=0}^{n} C(n,j) x^{n-j} y^j$

$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$

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Alternate proof:

$(x+y)^n \;=\; (x+y)(x+y)\cdots(x+y)$ n terms.

Result — choose an x or a y from each term.

How many ways to get $x^{n-j} y^j$?

Choose which terms give a y — $C(n,j)$.

Thus, $(x+y)^n = \sum_{j=0}^{n} C(n,j) x^{n-j} y^j$ for $n \geq 1$. $\square$

Using the binomial theorem

Thm. $n \geq 1$. $\sum_{k=0}^{n} C(n,k) = 2^n$.

Pf. $2^n = (1+1)^n = \sum_{k=0}^{n} C(n,k) 1^{n-k} 1^k = \sum_{k=0}^{n} C(n,k)$ $\square$

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Thm. $n \geq 1$. $\sum_{k=0}^{n} (-1)^k C(n,k) = 0$.

Pf. $0 = (1+(-1))^n = \sum_{k=0}^{n} C(n,k) 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k C(n,k)$ $\square$

Combinations with repetition

Example

Experiment: choose balls from: 3 red. 4 green. 5 blue.

How many ways to get 1 red ball and 2 green balls? $3 \cdot 4^2$

Note: not interested in order.

Represent result as: $n_1 \mid n_2 \mid n_3$, where

n_1 — number of red balls

n_2 — number of green balls

n_3 — number of blue balls