Algorithms and Probability

Lecture 3

September 14, 2020

Choosing with repetition

Example

Experiment: choose balls from: 3 red. 4 green. 5 blue.

How many ways to get 1 red ball and 2 green balls? $3 \cdot 4^2$

Note: not interested in order, but in which balls are chosen.

Combinations with repetition

Example

Experiment: choose r balls from: red, green, blue.

Represent result as: $n_1 \mid n_2 \mid n_3$, where

n_1 — number of red balls

n_2 — number of green balls

n_3 — number of blue balls

Combinations with repetition

Example: $r = 3$

Consider: $1 \mid 2 \mid 0$.

Could also represent as $*|**|$.

How many distinct results are there if 3 balls chosen? $(n_1 \mid n_2 \mid n_3)$

Combinations with repetition

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How many distinct results are there if 3 balls chosen? $(n_1 \mid n_2 \mid n_3)$

The number of ways to write down 3 *s and 2 $|$s.

(For example, $*| * *|$ or $| * * * |$.)

There are 5 places. Choose 3 places for stars.

Answer: $\binom{5}{3} = \frac{5!}{3!2!} = 10$.

Combinations with repetition

Example

Same experiment with n types of balls and r choices.

(An r-combination from n elements, with repetition.)

To figure out how many distinct results, how many stars do we use and how many bars. (Poll)

Combinations with repetition

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To figure out how many distinct results, how many stars do we use and how many bars. (Poll)

There are r stars and $n - 1$ bars.

Number of distinct results is $\binom{n+r-1}{r}$.

Combinations with repetition

Example

How many ways can you choose k integers

≥ 0, $(n_1, n_2, ..., n_k)$, such that $\sum_{i=1}^{k} n_i \;=\; n$?

Combinations with repetition

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$$\geq 0,\ (n_1, n_2, ..., n_k), \text{ such that } \sum_{i=1}^{k} n_i \ = \ n?$$

Answer: $\binom{k+n-1}{n}$.

Combinations with repetition

Example

How many nonnegative solutions are there to

$$n_1 + n_2 + n_3 + n_4 = 7?$$

Answer: $\binom{4+7-1}{7} = \frac{10!}{7!3!} = 120.$

Combinations with repetition

Example

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Are we considering $1 + 1 + 2 + 3$ to be different from $1 + 2 + 3 + 1$? (Poll)

Combinations with repetition

A bag with 3 red balls, 4 green balls, and 5 blue balls.

Consider all red balls the same, etc.

How many orderings are there of the balls?

Combinations with repetition

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How many orderings are there of the balls?

There are (3+4+5)! orderings.

Given one ordering, $(x_1, x_2, \ldots, x_{12})$,

there are 3!4!5! which are identical to it.

The number of distinct orderings is $\frac{12!}{3!4!5!} = 27,720$.

Combinations with repetition

Example

How many ways to put n pigeons in k holes,

if the ith hole should have n_i pigeons?

Answer: $\binom{n}{n_1}$ for the first hole.
$\binom{n-n_1}{n_2}$ for the second hole.
$\binom{n-n_1-n_2}{n_3}$ for the third hole.

Etc. Multiply these together to get:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Generating permutations

Suppose you want to test your sorting algorithm.

Try it on all permutations of $\{1, 2, ..., n\}$.

How large should n be?

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Lexicographic order would be nice.

$(2, 4, 7, 3, 8, 1, 9, 6, 5, 10) < (2, 4, 7, 5, 6, 10, 9, 1, 3, 8)$ because $3 < 5$.

How to get the next larger permutation? (after $(a_1, a_2, ..., a_n)$)

(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1).

Generating permutations

(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1).

If $a_{n-1} < a_n$, switch them.

Otherwise, find largest j with $a_j < a_{j+1}$.

$a_{j+1} > a_{j+2} > ... > a_n$.

Find the least value $a_k > a_j$ from $\{a_{j+1}, a_{j+2}, ..., a_n\}$.

Put a_k where a_j was.

Put the remaining elements from $\{a_j, a_{j+1}, ..., a_n\}$ in increasing order.

$(1, 2, 5, 10, 7, \mathbf{4}, 9, 8, 6, 3) \rightarrow$

$(1, 2, 5, 10, 7, \mathbf{6}, 3, 4, 8, 9)$.

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$(1, 2, 5, 10, 7, \mathbf{4}, 9, 8, 6, 3) \to$

$(1, 2, 5, 10, 7, \mathbf{6}, 3, 4, 8, 9)$.

(Poll)

Generating permutations

procedure next_permutation($a_1, a_2, ..., a_n$)
{ a permutation of $(1, 2, ..., n), \neq (n, n-1, ..., 1)$}
$j \leftarrow n - 1$
while $a_j > a_{j+1}$; $j \leftarrow j - 1$
{ j is largest subscript with $a_j < a_{j+1}$}

$k \leftarrow n$
while $a_j > a_k$; $k \leftarrow k - 1$
{ a_k is smallest value $> a_j$ to right of a_j}

switch a_j and a_k
$r \leftarrow n$
$s \leftarrow j + 1$
while $r > s$
    switch a_r and a_s
    $r \leftarrow r - 1$; $s \leftarrow s + 1$
{ this reverses the order after a_j}

Generating combinations

Combinations are subsets:

So use binary strings to represent them.

Lexicographic order of strings is increasing order of integers.

$(000), (001), (010), (011), (100), (101), (110), (111)$.

To get next integer, find rightmost 0.

Change it to 1. Change all 1's to right to 0's.

Generating combinations

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$(000), (001), (010), (011), (100), (101), (110), (111).$

To get next integer, find rightmost 0.

Change it to 1. Change all 1's to right to 0's.

For an r-combination, have r ones in the string.

$(00111), (01011), (01101), (01110), (10011), (10101), (10110),$

$(11001), (11010), (11100).$

Find the rightmost 01.

Change it to 10.

Move all the 1's to the right as far right as possible.

Generating combinations

To find the next r-combination of $(1, 2, ..., n)$:

$(11100) \leftrightarrow (1, 2, 3)$ — the smallest.

Lexicographic order is not lexicographic order of strings.

To get the next, find the rightmost 10.

Change it to 01.

Move all the 1's to the right as far left as possible.

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To do this directly:

Suppose have $(a_1, a_2, ..., a_r)$.

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Suppose have $(a_1, a_2, \ldots, a_r)$.

If this looks like $(a_1, \ldots, a_k, n-r+k+1, n-r+k+2, \ldots, n-1, n)$,

should change a_k to $a_k + 1$.

(in what follows, use the old value of a_k)

$a_{k+1} \leftarrow a_k + 2$.

$a_{k+2} \leftarrow a_k + 3$.

Generally, $a_j = a_k + j - k + 1$, for $k + 1 \leq j \leq r$.

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Generally, $a_j = a_k + j - k + 1$, for $k + 1 \leq j \leq r$.

Suppose $n = 6$. Consider $(2, 5, 6) \rightarrow (3, 4, 5)$.

Generating combinations

procedure next_combination($a_1, a_2, ..., a_r$)
{ a r-combination of $(1, 2, ..., n)$, $\neq (n - r + 1, ..., n)$}
$i \leftarrow r$
while $a_i = n - r + i$
    $i \leftarrow i - 1$
$a_i \leftarrow a_i + 1$
for $j \leftarrow i + 1$ **to** r
    $a_j \leftarrow a_i + j - i$

Discrete Probability

Example: 6-sided dice — 2

experiment — throwing the dice, getting 2 numbers

sample space — all pairs of numbers (i, j), where $1 \leq i, j \leq 6$.

event — pairs of numbers that sum to 7

probability that the sum is 7 — $\frac{6}{6 \cdot 6} = \frac{1}{6} = \frac{|E|}{|S|}$,

where E is an event and S is a finite sample space.

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Example: Deck of 52 playing cards:

experiment — taking 13 different cards at random

sample space — all hands containing 13 cards

event — all hands with no face cards or aces

probability — $\frac{\frac{36!}{13!23!}}{\frac{52!}{13!39!}} = \frac{36!39!}{23!52!} = .00363896\ldots$

Discrete Probability

Example: 36 numbered balls

experiment — randomly choosing 7 balls and then 4 more

sample space — all pairs of sets of numbers (A, B), where all numbers are distinct, $|A| = 7$, $|B| = 4$.

event — a specific set of pairs $(A_1, B_1), \ldots, (A_k, B_k)$, such that each A_i contains 6 of the 7 distinct numbers $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and each B_i contains the number missing from A_i.

probability is $\frac{k}{\frac{36!}{7!29!} \cdot \frac{29!}{4! \cdot 25!}} = \frac{k}{\frac{36!}{4!7!25!}}$.

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So what is k? There are 7 possibilities for which x_j is not in A_i, 29 possibilities for that last number in A_i, and $28!/(3!25!)$ possibilities for the extra elements in B_i.

So $k = 7 \cdot 29 \cdot 28!/(3!25!)$,

and the entire probability is $\frac{7 \cdot 29!/(3!25!)}{\frac{36!}{4!7!25!}} = \frac{4 \cdot 7 \cdot 29!7!}{36!} < 3.3542 * 10^{-6}$.