DM551/MM851

Algorithms and Probability

September 28, 2020

Expectations

Recall:

Def. A *random variable* is a function $f : S \to \mathbb{R}$.

Def. For a finite sample space $S = \{s_1, s_2, ..., s_n\}$, the *expected value* of the random variable $X(s)$ is

$$E(X) = \sum_{i=1}^{n} p(s_i)X(s_i).$$

Def. For a countably infinite sample space $S = \{s_i \mid i \geq 1\}$, the *expected value* of the random variable $X(s)$ is
$E(X) = \sum_{i=1}^{\infty} p(s_i)X(s_i)$.

Expectations

Recall:

Def. A *random variable* is a function $f : S \to \mathbb{R}$.

Def. For a finite sample space $S = \{s_1, s_2, ..., s_n\}$, the *expected value* of the random variable $X(s)$ is

$$E(X) = \sum_{i=1}^{n} p(s_i)X(s_i).$$

Def. For a countably infinite sample space $S = \{s_i \mid i \geq 1\}$, the *expected value* of the random variable $X(s)$ is $E(X) = \sum_{i=1}^{\infty} p(s_i)X(s_i)$.

Example: What is the expected number of successes in n Bernoulli trials? Probability of success $= p$. Probability of failure $= q = 1 - p$.

Expectations

Answer:

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{n} k \cdot p(X = k) \\
&= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \left(\frac{n!}{k!(n-k)!} \right) p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \left(\frac{n(n-1)!}{k(k-1)!(n-1-k+1)!} \right) p^k q^{n-k} \\
&= n \sum_{k=1}^{n} \left(\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} \right) p^k q^{n-k} \\
&= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{n-k}
\end{aligned}
$$

Expectations

$$\begin{aligned}
E[X] &= n\sum_{k=1}^{n}\binom{n-1}{k-1}p^k q^{n-k} \\
&= np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1} q^{n-k} \\
&= np\sum_{j=0}^{n-1}\binom{n-1}{j}p^{j} q^{n-1-j} \\
&= np(p+q)^{n-1} \\
&= np. \qquad \square
\end{aligned}$$

Expectations

Example: What is the expected value of the first successful Bernoulli trial?

Answer:

$$
\begin{aligned}
\sum_{i=1}^{\infty} iq^{i-1}p &= \sum_{i=1}^{\infty} iq^{i-1} - \sum_{i=1}^{\infty} iq^{i} \\
&= \sum_{j=0}^{\infty} (j+1)q^{j} - \sum_{i=1}^{\infty} iq^{i} && (j = i-1) \\
&= 1 + \sum_{j=1}^{\infty} (j+1-j)q^{j} && . \\
&= 1 + \sum_{j=1}^{\infty} q^{j} \;=\; 1 + (\sum_{j=0}^{\infty} q^{j}) - 1 \\
&= \frac{1}{1-q} \\
&= \frac{1}{p}
\end{aligned}
$$

Expectations

Example: What is the expected value of the first successful Bernoulli trial?

Answer:

$$\begin{aligned}
\sum_{i=1}^{\infty} iq^{i-1}p &= \sum_{i=1}^{\infty} iq^{i-1} - \sum_{i=1}^{\infty} iq^{i} \\
&= \sum_{j=0}^{\infty} (j+1)q^{j} - \sum_{i=1}^{\infty} iq^{i} \qquad\qquad (j = i-1) \\
&= 1 + \sum_{j=1}^{\infty} (j+1-j)q^{j} \qquad\qquad . \\
&= 1 + \sum_{j=1}^{\infty} q^{j} \;=\; 1 + (\sum_{j=0}^{\infty} q^{j}) - 1 \\
&= \tfrac{1}{1-q} \\
&= \tfrac{1}{p}
\end{aligned}$$

With a fair die, the expected number of throws before a 1 is 6.

Linearity of Expectations

A *linear function* has the form

$$f(X_1, X_2, ..., X_n) = a_0 + a_1 X_1 + a_2 X_2 + ... + a_n X_n$$

where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$.

Linearity of Expectations

A *linear function* has the form

$$f(X_1, X_2, ..., X_n) = a_0 + a_1 X_1 + a_2 X_2 + ... + a_n X_n$$

where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$.

Thm. Let f be a linear function, S be a sample space, and $X_1, X_2, ..., X_n$ be random variables defined on S. Then, $E[f(X_1, X_2, ..., X_n)] = f(E[X_1], E[X_2], ..., E[X_n])$.

Linearity of Expectations

Pf. Let $f(X_1, \ldots, X_n) = a_0 + a_1 X_1 + \ldots + a_n X_n$ where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$. Then,

$$
\begin{aligned}
E[f(X_1, \ldots, X_n)] &= \sum_{s \in S} p(s) f(X_1(s), \ldots, X_n(s)) \\
&= \sum_{s \in S} p(s)(a_0 + a_1 X_1(s) + \ldots + a_n X_n(s)) \\
&= \sum_{s \in S} \left(p(s) a_0 + \sum_{i=1}^{n} p(s) a_i X_i(s) \right) \\
&= a_0 + \sum_{i=1}^{n} \left(\sum_{s \in S} p(s) a_i X_i(s) \right) \\
&= a_0 + \sum_{i=1}^{n} \left(a_i \sum_{s \in S} p(s) X_i(s) \right) \\
&= f(E[X_1], E[X_2], \ldots, E[X_n]). \qquad \square
\end{aligned}
$$

Example: Hatcheck problem

An employee collects n hats from customers in a restaurant.

Hats are returned randomly. What is the expected number of customers who get their own hat? (Poll)

Linear Search Algorithm

procedure linear_search($x, a_1, a_2, ..., a_n$)
$i \leftarrow 1$
while ($i \leq n$ and $x \neq a_i$) $i \leftarrow i + 1$
{ Either $i = n + 1$ or $x = a_i$}

if $i \leq n$ **then** *location* $\leftarrow i$
else *location* $\leftarrow 0$
return *location*

Linear Search Algorithm

procedure linear_search($x, a_1, a_2, ..., a_n$)
$i \leftarrow 1$
while ($i \leq n$ and $x \neq a_i$) $i \leftarrow i + 1$
{ Either $i = n + 1$ or $x = a_i$}

if $i \leq n$ **then** *location* $\leftarrow i$
else *location* $\leftarrow 0$
return *location*

Worst case: n comparisons of elements ($2n + 1$ comparisions).
(Poll)

Insertion Sort Algorithm

procedure InsertionSort(List):
{ Input: List is a list }
{ Output: List, with same entries, but in nondecreasing order }

$N := 2$
while ($N \leq$ length(List)
  Pivot := Nth entry
  $j := N - 1$
  while ($j > 0$ and jth entry $>$ Pivot)
    move jth entry to loc. $j + 1$
    $j := j - 1$
  place Pivot in $j + 1$st loc.
  $N := N + 1$

Insertion Sort Algorithm

procedure InsertionSort(List):
{ Input: List is a list }
{ Output: List, with same entries, but in nondecreasing order }

```
N := 2
while (N ≤ length(List)
    Pivot := Nth entry
    j := N − 1
    while (j > 0 and jth entry > Pivot)
        move jth entry to loc. j + 1
        j := j − 1
    place Pivot in j + 1st loc.
    N := N + 1
```

Worst case: $\sum_{i=2}^{n}(i-1) = \frac{n(n-1)}{2}$ comparisons of elements (Poll)

Expectation, Variance, Standard Deviation

If two random variables X and Y are independent, then $E[XY] = E[X] \cdot E[Y]$.

Expectation, Variance, Standard Deviation

If two random variables X and Y are independent, then $E[XY] = E[X] \cdot E[Y]$.

The **variance** of a random variable is $V[X] = E[(X - E[X])^2]$.

$V[X] = E[X^2 - 2XE[X] + E^2[X]]$.

By the linearity of expectations, this is $E[X^2] - 2E[XE[X]] + E[E^2[X]]$.

Since $E[X]$ is a real number, this is $E[X^2] - 2E^2[X] + E^2[X]$.

Thus, $V[X]$ is also $E[X^2] - E^2[X]$.

Expectation, Variance, Standard Deviation

If two random variables X and Y are independent, then $E[XY] = E[X] \cdot E[Y]$.

The **variance** of a random variable is $V[X] = E[(X - E[X])^2]$.

$V[X] = E[X^2 - 2XE[X] + E^2[X]]$.

By the linearity of expectations, this is $E[X^2] - 2E[XE[X]] + E[E^2[X]]$.

Since $E[X]$ is a real number, this is $E[X^2] - 2E^2[X] + E^2[X]$.

Thus, $V[X]$ is also $E[X^2] - E^2[X]$.

If X and Y are independent random variables, then $V[X + Y] = V[X] + V[Y]$. If $X_1, X_2, ..., X_n$ are pairwise independent random variables, then $V[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} V[X_i]$.

The **standard deviation** of a random variable is the positive square root of the variance.