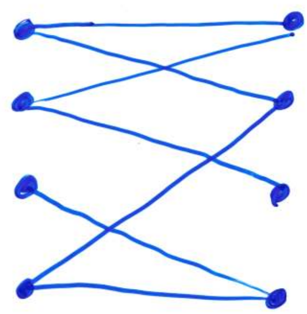
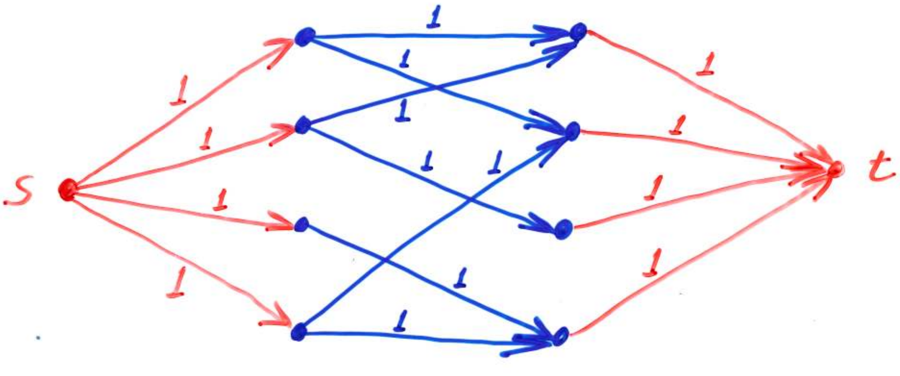
Bipartite Matching Problem

- Cardinality matching

Given a bipartite graph, find as large a matching as possible.
(No weights, or all weights = 1.)





Each matching $\Rightarrow$ flow
Each integral flow $\Rightarrow$ matching

$\therefore$ Can compute a maximum matching in time $O(|E|^{3/2})$
(or even better $O(|E|\sqrt{|V|})$).
$O(|V||E|)$ from Ford Fulkerson

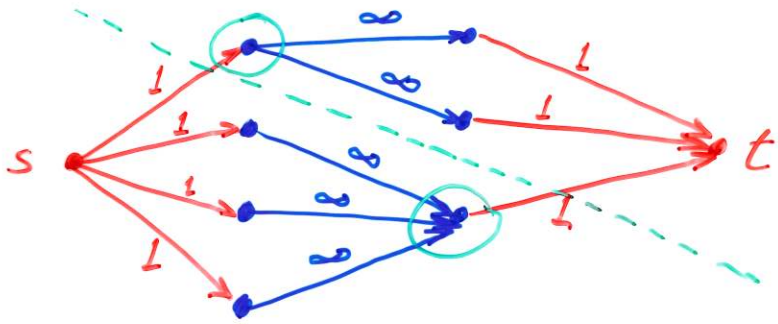
Def $C = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ is a **cover** if every edge is incident with some vertex of C.

Note For any matching M and any cover C, $|M| \le |C|$.

(because, for every edge in M, any cover must include at least one of the 2 endpoints.)

König-Egervary Thm $\max_M |M| = \min_C |C|$

Pf



A min-cut doesn't include any edges from the original graph. For the cover, choose those vertices ($\neq s,t$) incident to the edge across the cut. Suppose some edge was not covered by this. It would be in the cut. $\cancel{\to}$

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What about the dual to the linear program for the weighted matching problem?

Suppose the bipartite graph is given by an adjacency matrix.

$$
\begin{array}{c|cccccc|}
 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
\hline
x_1 & & & x & x & x & \\
x_2 & & x & & & & \\
x_3 & x & & & & & \\
x_4 & & x & & & & \\
x_5 & & x & & & & \\
x_6 & & x & & & & \\
\hline
\end{array}
$$

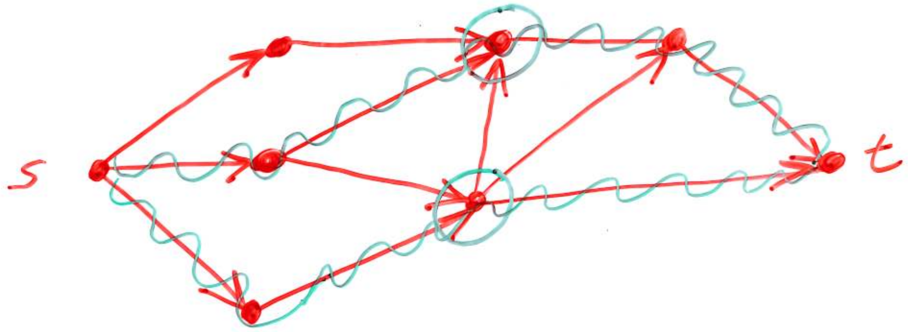
Matching - a set of independent x's, no 2 on the same line (row or column)

Cover - a set of lines that cover all the x's.

König-Egervary Thm - The maximum # of independent x's is = the minimum # of lines to cover all x's.

Other Min-Max Theorems

__Menger's Thm__ If a digraph G is k-vertex-connected from s to t and does not contain an edge from s to t, then G contains k independent directed paths from s to t.



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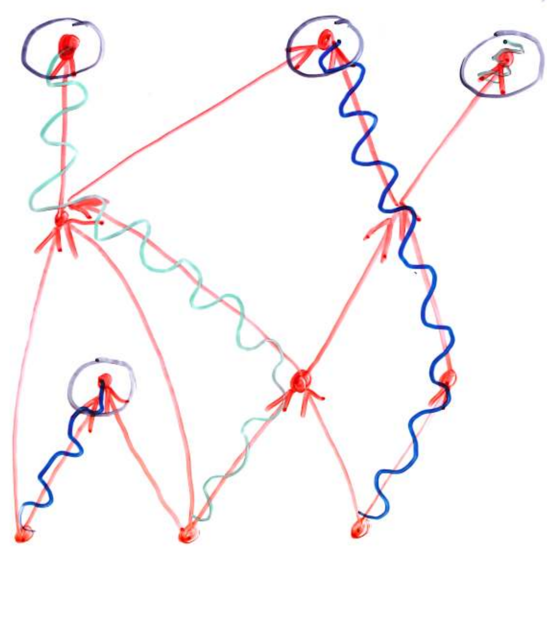
__Dilworth's Thm__ For a given partial order $(S, \leq)$, a chain of elements is a sequence $s_1, s_2, \ldots, s_m$ where $s_i \leq s_{i+1}$. The minimum number of chains, s.t. every element is contained in at least one chain, is equal to the maximum number of incomparable elements.



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