Online Algorithms with Advice

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Overview

Section 1

The advice model

Standard online algorithm model

Input $I = \langle x_1, \ldots, x_n \rangle$ → Online algorithm → Output $O = \langle y_1, \ldots, y_n \rangle$

Adversary

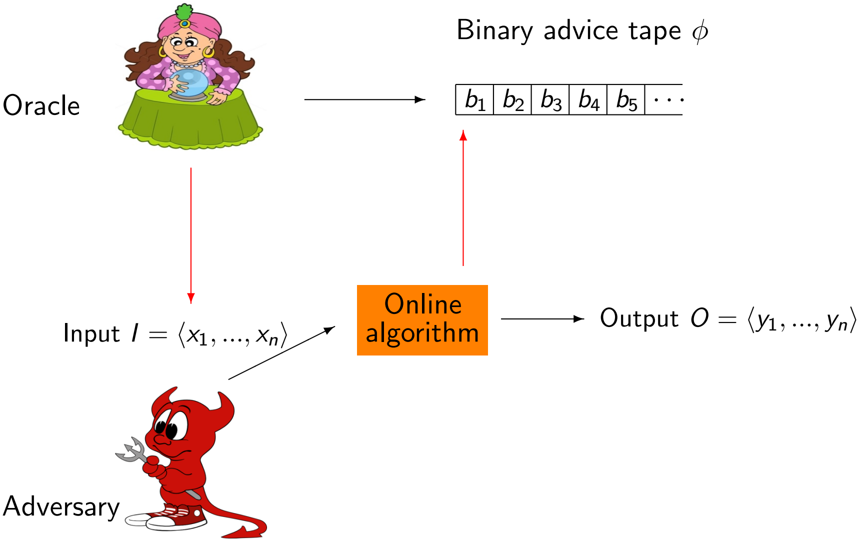
Competitive Analysis

Compare the performance of an online algorithm, ALG, with an optimal **offline** algorithm, OPT:

- OPT knows the whole sequence in the beginning.

Competitive ratio of ALG is the maximum ratio between the cost of ALG and OPT for serving the same sequence (minimization problems).

Advice from an Oracle



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- There are other advice models for bin packing
  - Original: [Dobrev, Královič, Markou, 2009]
  - Advice with request: [Fraigniaud,Korman,Rosén, 2011]

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- a measure of how much knowledge of the future an online algorithm needs to achieve a certain competitive ratio
- a problem-independent and quantitative approach
- a means of modelling keeping parallel solutions, multi-solution model
- sometimes useful in practice

Relevant Questions

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[Dobrev,Královič,Pardubská, 2009]
The advice complexity for optimality is 1 bit:

$$b = \begin{cases} 0 & \text{if } |L| < d \\ 1 & \text{if } |L| \geq d \end{cases}$$

Optimality example: Paging

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Asymptotically, for large k, the advice complexity for optimality is 1 bit per request:

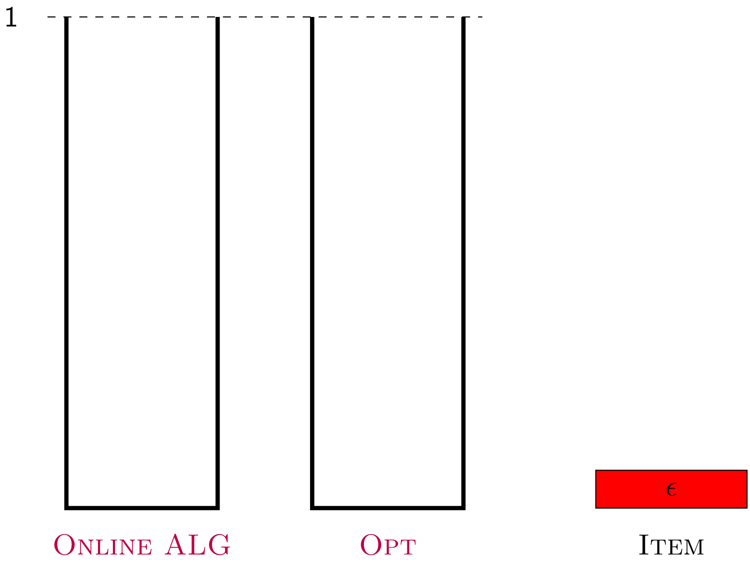
$$b_i = \begin{cases} 0 & \text{if } \textsc{Opt} \text{ would have } r_i \text{ in cache next time it is requested} \\ 1 & \text{otherwise} \end{cases}$$

Competitive ratio example: Simple knapsack problem

Simple knapsack problem:

- Knapsack of size 1
- Items of size $\in (0, 1]$ arrive online
- An item must be accepted or rejected
- Goal maximize total size accepted (total size ≤ 1)

Competitive ratio example: Simple knapsack problem



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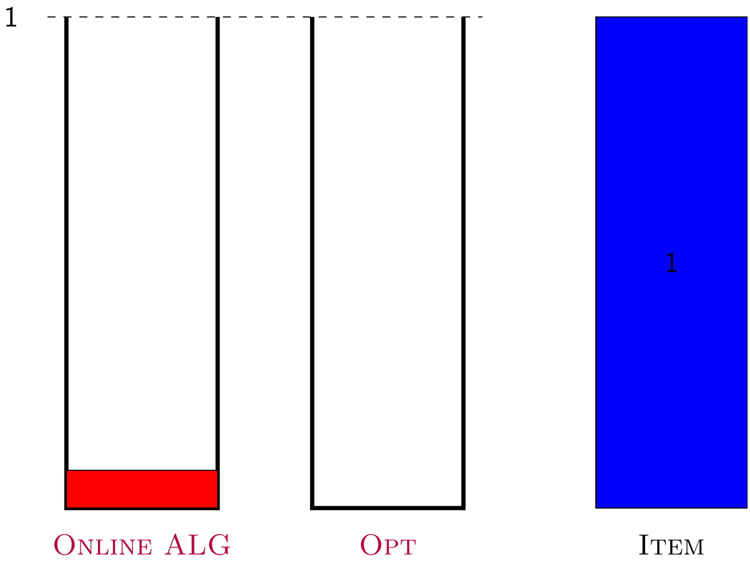

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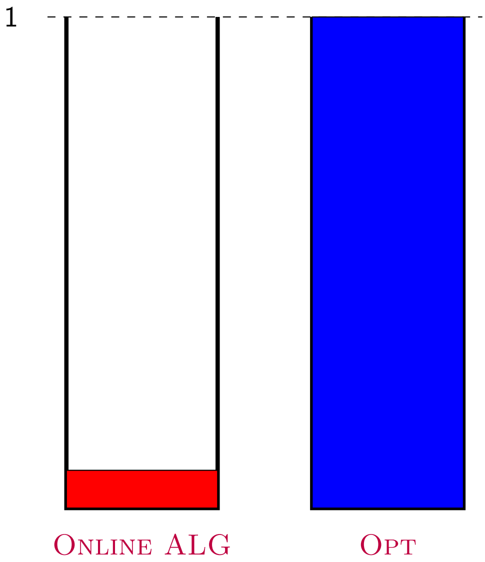

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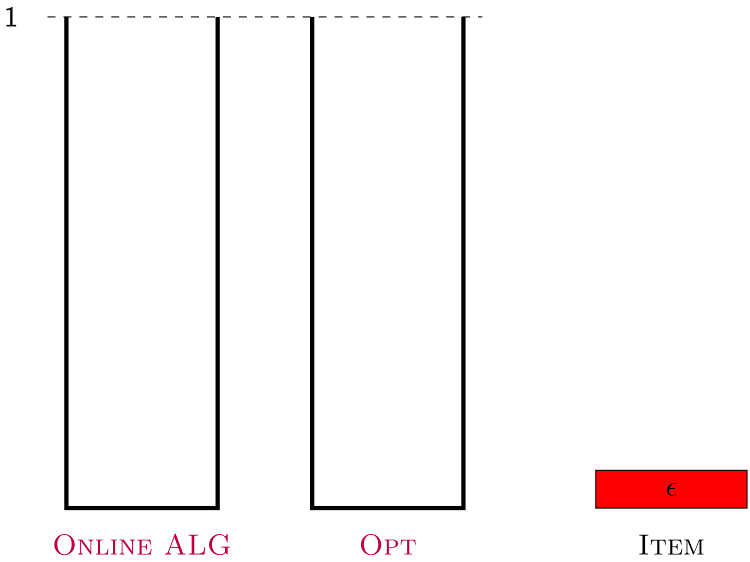

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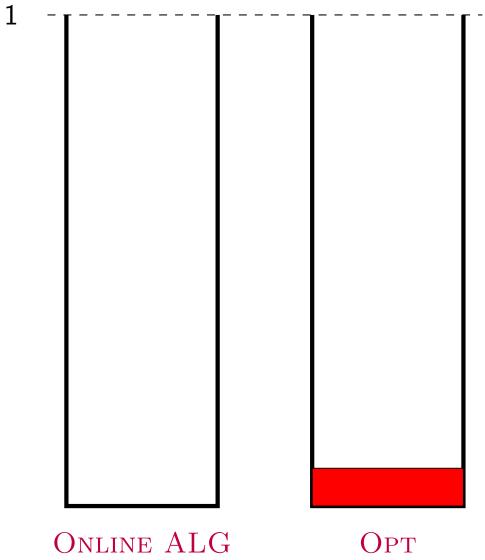

No more items

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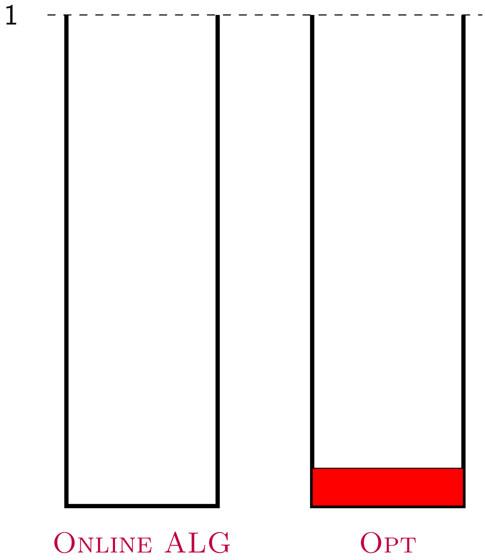

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Algorithm:

- If $b = 0$, use Greedy.
- If $b = 1$, wait for item of size $> \frac{1}{2}$.

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- If $b = 0$, use Greedy.
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Fact: If there is no item of size $> \frac{1}{2}$, Greedy will accept at least

$$\min\{\mathrm{OPT}(I), 1/2\}$$

(in order to reject anything, it must have already accepted $\frac{1}{2}$).

Section 2

The bin packing problem

Bin Packing Problem

Input: items of various sizes $\in (0, 1]$

Output: packing of all items into unit size bins

Goal: use minimum number of bins

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Output: packing of all items into unit size bins

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Applications: storage, cutting stock...

Offline Bin Packing Problem

The problem is NP-hard: Reduce from 2-PARTITION.

First-Fit-Decreasing has an approximation ratio of $11/9 \approx 1.22$ [Johnson,Demers,Ullman,Garey,Graham, 1974]

There is an asymptotic PTAS for the problem [de la Vega,Lueker, 1981]

Online Bin Packing Problem

Request sequence is revealed in a sequential, online manner.

Examples:

- Next-Fit
- First-Fit
- Best-Fit
- Harmonic, Harmonic++
- Heydrich, van Stee

First-Fit vs. Next-Fit — Online

First-Fit

- Find the first open bin with enough space, and place the item there
- If such a bin does not exist, open a new bin

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Next-Fit

- Put item in current open bit, if it fits
- Otherwise, close that bin and open a new current bin

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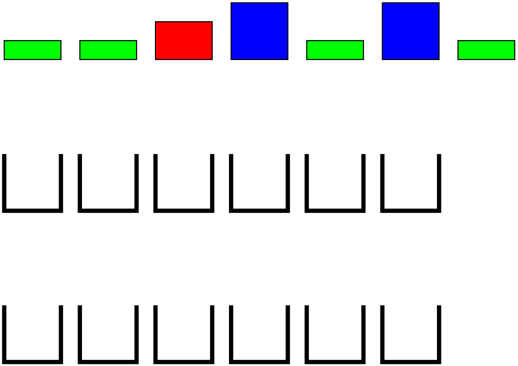

First-Fit

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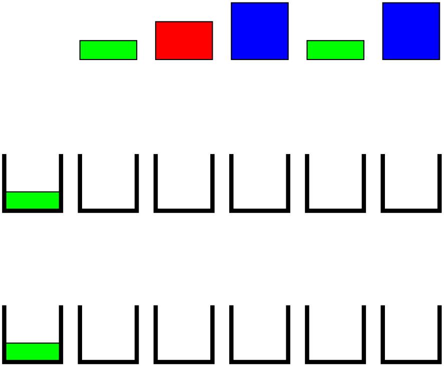

FIRST-FIT

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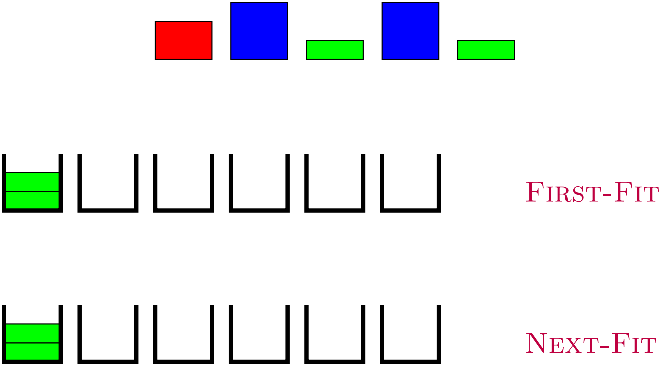

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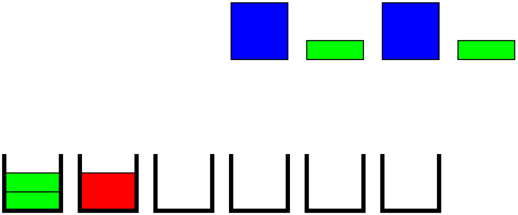

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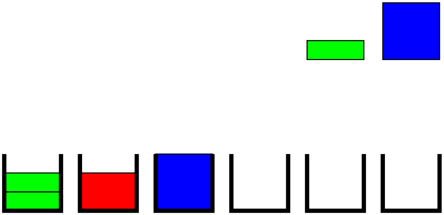

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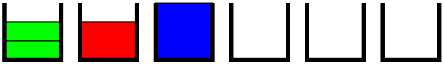

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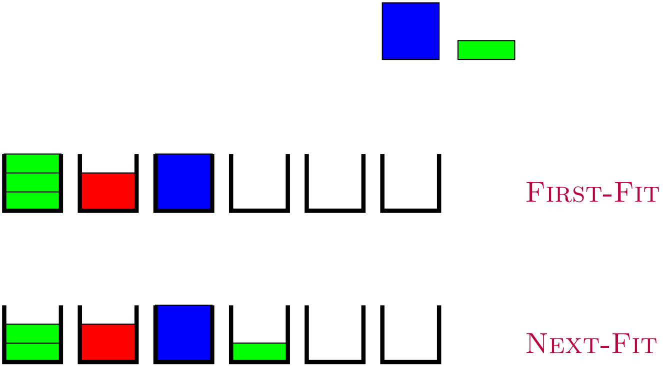

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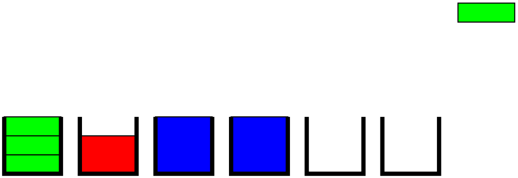

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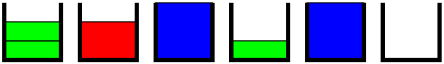

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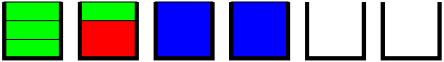

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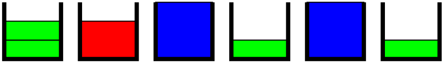

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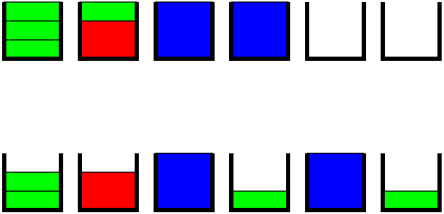

First-Fit Result: 4

Next-Fit Result: 6

Competitive Analysis

Next-Fit has competitive ratio 2
[Johnson, 1974]

Best-Fit and First-Fit have competitive ratio 1.7
[Johnson,Demers,Ullman,Garey,Graham, 1974]

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Recall that offline First-Fit-Decreasing has approximation ratio ≈ 1.22.

- A big gap between quality of online and offline solutions.
- What about an "almost online" algorithm? What about advice?

Relevant Questions

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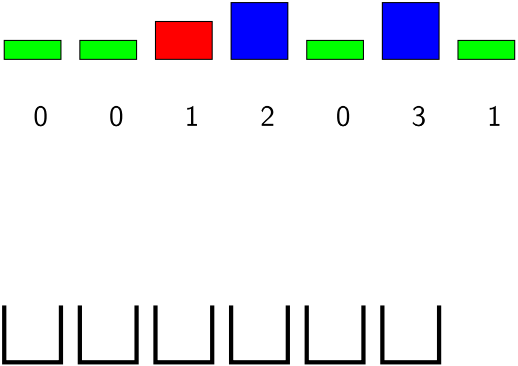
Most advice results are from [B.,Kamali,Larsen,López-Ortiz, 2016]

Optimal Solution with Advice

How many bits of advice are sufficient to achieve an optimal solution?

- Advice for each item: index of target bin in $\textsc{Opt}$'s packing.
- $n\lceil \log \textsc{Opt}(\sigma) \rceil$ bits of advice are sufficient



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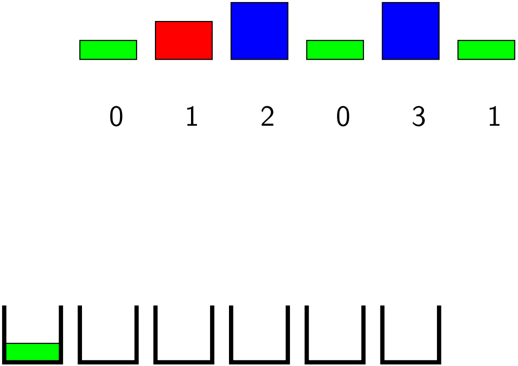

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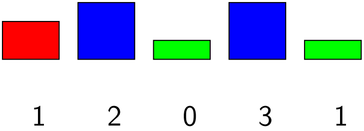

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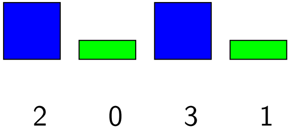

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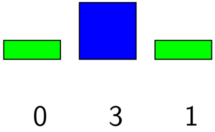

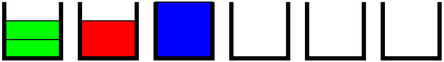

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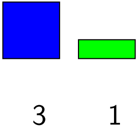

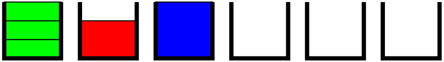

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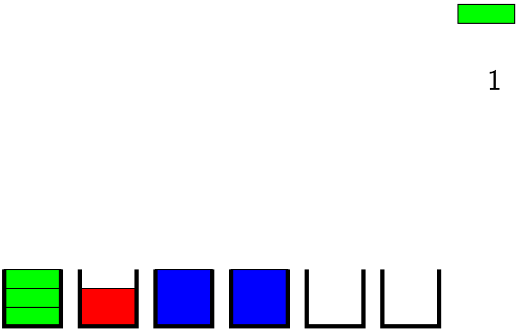

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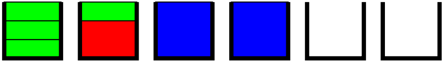

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Comparison:
$n\lceil \log \mathbf{Opt}(\sigma) \rceil$ bits of advice are sufficient for optimality.
$(n - 2\,\mathbf{Opt}(\sigma))\log \mathbf{Opt}(\sigma)$ bits of advice are required to guarantee optimality.

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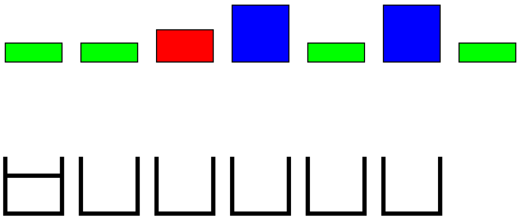
Algorithm: Reserve a space of size $2/3$ for each of them
Apply First-Fit for the other items.

Advice: 1



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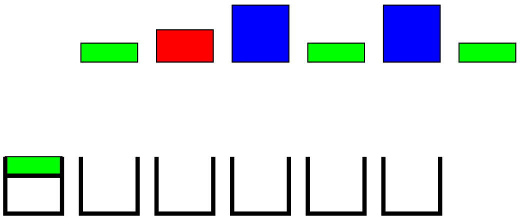

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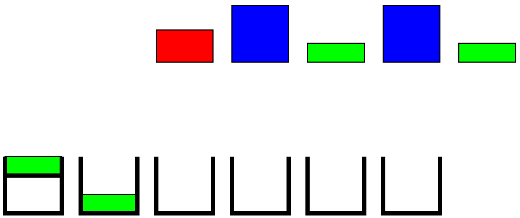

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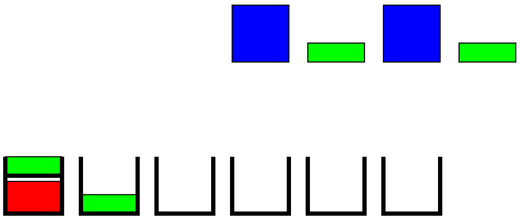

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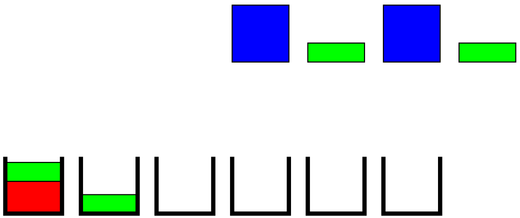

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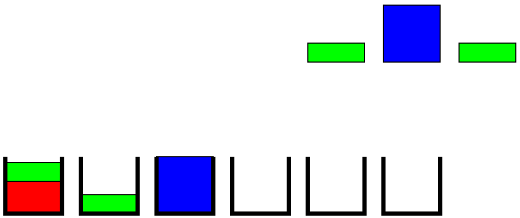

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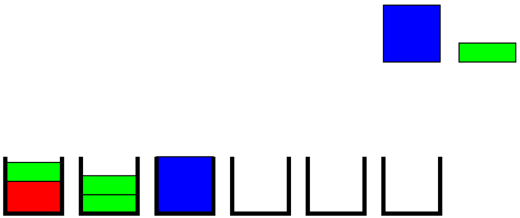

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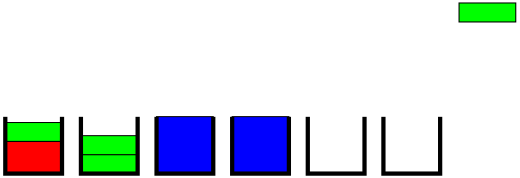

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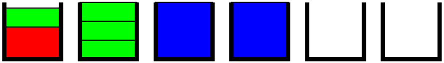

Breaking the Lower Bound — With Less Advice

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Newer result: [Angelopoulos,Dürr,Kamali,Renault,Rosén, 2018]

With constant advice, one can get a competitive ratio arbitrarily close to 1.47012.

Unfortunately, the advice depends on $\textsc{Opt}$.

Advice of Linear Size

An online algorithm which receives 2 bits of advice per request (plus an additive lower order term).

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Advice depends on $\textsc{Opt}$'s packing.

Advice of Linear Size

One can obtain results similar to PTAS results:

[Renault,Rósen,van Stee, 2015]

Theorem

There is an online bin packing algorithm which is $(1+3\delta)$-competitive (or asymptotically $(1+2\delta)$-competitive), using $s = \frac{1}{\delta}\log\frac{2}{\delta^2} + \log\frac{2}{\delta^2} + 3$ bits of advice per request.

A Lower Bound

A linear amount of advice is required to achieve a competitive ratio better than $9/8$.

Get a trade-off — better ratio requires more advice

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Reduction order:

Binary string guessing problem $\longrightarrow$ Binary separation problem

Binary separation problem $\longrightarrow$ Bin packing problem

Binary string guessing problem (with known history): 2-SGKH
[Emek,Fraigniaud,Korman,Rosén, 2011]
[Böckenhauer,Hromkovic,Komm,Krug,Smula,Sprock, 2014]

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- $\langle 0,\ 1,\ 0,\ ?\rangle$
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Theorem

On inputs of length n, any deterministic algorithm for 2-SGKH that is guaranteed to guess correctly on more than αn bits, for $1/2 \leq \alpha < 1$, needs to read at least $(1 + (1-\alpha)\log(1-\alpha) + \alpha\log(\alpha))n$ bits of advice.

Note: If we assume the number, n_0, of 0s is given, we need at least $(1 + (1-\alpha)\log(1-\alpha) + \alpha\log(\alpha))n - e(n_0)$ bits of advice, where $e(n_0) = \lceil\log(n_0+1)\rceil + 2\lceil\log(\lceil\log(n_0+1)\rceil + 1)\rceil$ (self-delimiting code).

Binary Separation Problem

Binary separation problem:

- For a sequence of $n_1 + n_2$ items decide whether an item belongs to the n_1 smaller items or n_2 larger items.

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- Don't have to choose in $[0, 1]$.
- Don't have to choose the exact middle value.

Reduction from 2-SGKH to Binary separation

```
small = 0; large = 1
repeat
   mid = (large + small) / 2
   class_guess = SeparationAlgorithm.ClassifyThis(mid)
   if class_guess = "large" then
      bit_guess = 0
   else
      bit_guess = 1
   actual_bit = Guess(bit_guess)
   {The actual value is received after guessing (2-SGKH).}
   if actual_bit = 0 then
      large = mid {We let "large" be the correct decision.}
   else
      small = mid {We let "small" be the correct decision.}
until end of sequence
```

Reduction from Binary separation to Bin packing

Idea: Create small and large items, so ALG has to decide which is which.

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Give large items, L and small items, S:

- OPT places large items with begin items.
- OPT places small items, one per bin.
- ALG much choose.

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For each small item of size $\frac{1}{2} - \epsilon_i$,
give an item of size $\frac{1}{2} + \epsilon_i$ — matching items, M.

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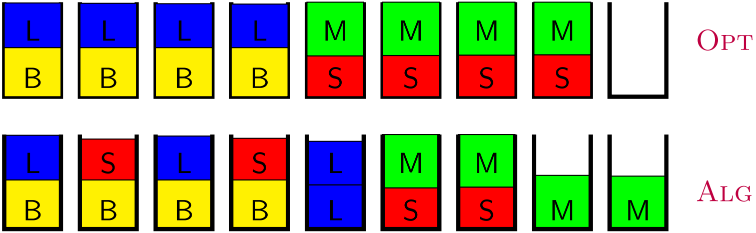
For each small item of size $\frac{1}{2} - \epsilon_i$,
give an item of size $\frac{1}{2} + \epsilon_i$ — matching items, M.
OPT packs matching items with small items, using $n_1 + n_2$ bins.

Reduction from Binary separation to Bin packing



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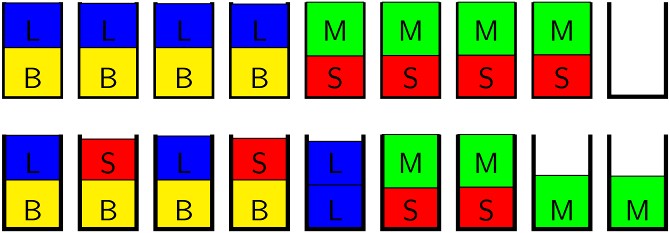

OPT Result: 8

ALG Result: 9

Reduction from Binary separation to Bin packing

Large item + matching item > 1.

Reduction from Binary separation to Bin packing

Large item + matching item $>$ 1.

Suppose large item is not with a begin item. Why?

- bad guess for that item
- bad guess for small item — no space

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At most 2 fit in a bin together.

4 errors in binary separation $\Rightarrow \geq 1$ more bin

Lower bound result for bin packing

Theorem

On inputs of length n, to achieve a competitive ratio of c ($1 < c < 9/8$), an online algorithm must get at least $(1 + (4c-4)\log(4c-4) + (5-4c)\log(5-4c))n - e(n)$ bits of advice.

Recall that $e(n) = \lceil \log(n+1) \rceil + 2\lceil \log(\lceil \log(n+1) \rceil + 1) \rceil$.

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On inputs of length n, to achieve a competitive ratio of c $(1 < c < 9/8)$*, an online algorithm must get at least*
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Newer result: [Angelopoulos,Dürr,Kamali,Renault,Rosén, 2016]
Can improve analysis from 4 mistakes causing at least 1 extra bin to 3 mistakes causing at least 1 extra bin.

Theorem

On inputs of length n, to achieve a competitive ratio of c $(1 < c < 7/6)$*, an online algorithm must get at least*
$(1 + (3c-3)\log(3c-3) + (4-3c)\log(4-3c))n - e(n)$ *bits of advice.*

Recall that $e(n) = \lceil \log(n+1) \rceil + 2\lceil \log(\lceil \log(n+1) \rceil + 1) \rceil$.

Lower bound result for bin packing

Even newer result: [Mikkelsen, 2016]
Can improve analysis using above result and using weighted binary string guessing.

Theorem

On inputs of length n, to achieve a competitive ratio of c ($1 < c < 4 - 2\sqrt{2}$), a randomized c-competitive bin packing algorithm must read at least $\Omega(n)$ bits of advice.

$9/8 = 1.125$, $7/6 \approx 1.1667$, $4 - 2\sqrt{2} \approx 1.1716$

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What does this say?

- One cannot get a competitive ratio better than 1.17 by giving information such as the number of bins that OPT uses.
- One cannot beat 1.17 by keeping $2^{o(n)}$ solutions in the multi-solution model.

Lower bound result for bin packing

[Mikkelsen, 2016]

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On inputs of length n, to achieve a competitive ratio of c ($1 < c < 4 - 2\sqrt{2}$), a randomized c-competitive bin packing algorithm must read at least $\Omega(n)$ bits of advice.

[Renault, Rosén, van Stee, 2015] For a fixed competitive ratio, there exists an online algorithm which only needs linear advice:
They present an algorithm for online bin packing which is $(1 + 3\delta)$-competitive (or asymptotically $(1 + 2\delta)$-competitive), using $s = \frac{1}{\delta} \log \frac{2}{\delta^2} + \log \frac{2}{\delta^2} + 3$ bits of advice per request.

Open Problems

- Linear advice is needed to be c-competitive, $c < 1.17$.
  Linear advice is sufficient for any fixed c. There is a huge gap, though.
- $(2 + o(1))n$ advice is sufficient to be $(4/3 + \epsilon)$-competitive.
  Can one get a better ratio with so few bits?
- Counting the number of items of size in $(1/2, 2/3]$
  ($O(\log n)$ bits of advice) is sufficient to be $3/2$-competitive.
  Are there other algorithms (based on advice) which could be practical?

Section 3

An advice complexity class, AOC

B.,Favrholdt,Kudahl,Mikkelsen, 2017

- Introduce an advice complexity class, Asymmetric Online Covering (AOC).
- Prove upper bound on advice complexity for all problems in AOC; advice complexity for competitive ratio c:

$$\frac{0.53n}{c} \leq \log_2\left(1 + \frac{(c-1)^{c-1}}{c^c}\right) n \leq \frac{n}{c}$$

- Many problems, including Vertex Cover, Independent Set, Set Cover, and Dominating Set are AOC-Complete.

Vertex Cover

Vertex Cover problem:

- Vertices of graph $G = (V, E)$ arrive online, with edges to previous vertices.
- Vertices must be accepted or rejected.
- Accepted vertices, $C \subseteq V$, must be a vertex cover, i.e. $\forall (u, v) \in E$, either u or $v \in C$.
- Goal: Minimize $|C|$.

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- Goal: Minimize $|C|$.

Note: Due to the vertex-arrival model, the 2-approximation algorithm which takes both endpoints of an uncovered edge, cannot be used.

Vertex Cover

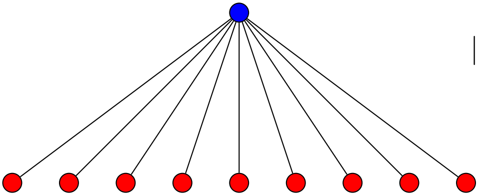
ALG

OPT

ALG must reject vertex; otherwise it will be the last.

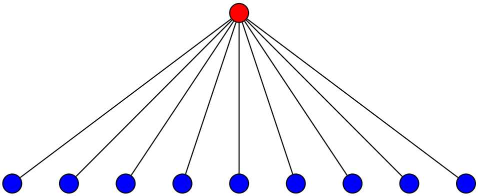
Vertex Cover

ALG



$|C| = n - 1$

OPT



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No.

How many bits do we need to be c-competitive?

Advice complexity of Vertex Cover

Let V_{OPT} be an optimal vertex cover.
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Oracle will specify which vertices to accept.
How many ck-subsets are needed to cover all k-subsets?

Advice complexity of Vertex Cover

Suppose $n = |V| = 6$, $c = 2$, and $|V_{\mathrm{OPT}}| = 2$.
There are $\binom{6}{2} = 15$ possibilities for V_{OPT}.

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These 3 subsets of size $ck = 4$ cover all 2-subsets of $\{1, 2, 3, 4, 5, 6\}$.

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2 advice bits are sufficient to specify one which covers V_{OPT}.

Covering designs

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By the Pigeonhole Principle,

$$\frac{\binom{n}{k}}{\binom{ck}{k}} \leq C(n, ck, k)$$

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[Erdös,Spencer, 1974]

$$\frac{\binom{n}{k}}{\binom{ck}{k}} \leq C(n, ck, k) \leq \frac{\binom{n}{k}}{\binom{ck}{k}} \left(1 + \ln \binom{ck}{k}\right)$$

Algorithm with advice for Vertex Cover

Oracle writes:

- values of n and k — $O(\log n)$ bits
- index of a ck-subset, C, in an (n, ck, k)-covering design, s.t. $V_{\mathrm{OPT}} \subseteq C$.

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ALG uses $C = \langle b_1, b_2, ..., b_n \rangle$:

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- if $b_i = 0$, reject v_i

ALG is c-competitive. It reads at most $B = \log \max_{k:ck<n} C(n, ck, k) + O(\log n)$ advice bits.

$$B \leq \log \left(1 + \frac{(c-1)^{c-1}}{c^c}\right) n + O(\log n) \leq \frac{n}{c} + O(\log n)$$

A class of online problems

Generalizing from the properties used for Vertex Cover, we define a class of online problems.

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Definition

An accept/reject minimization problem is in *Asymmetric Online Covering (AOC)* if

1. If Y is feasible, $\text{cost}(Y) = |Y|$. Otherwise $\text{cost}(Y) = \infty$.
2. A superset of a feasible solution is feasible.

Example problems in AOC:
Vertex Cover, Dominating Set, Set Cover, Cycle Finding.

A class of online problems

Generalizing from the properties used for Vertex Cover, we define a class of online problems.

It applies to *accept/reject* online problems: The online algorithm only accepts or rejects each request. Let the set of accepted requests be Y.

Definition

An accept/reject maximization problem is in *Asymmetric Online Covering (AOC)*, if

1. If Y is feasible, $\mathrm{cost}(Y) = |Y|$. Otherwise $\mathrm{cost}(Y) = -\infty$.
2. A subset of a feasible solution is feasible.

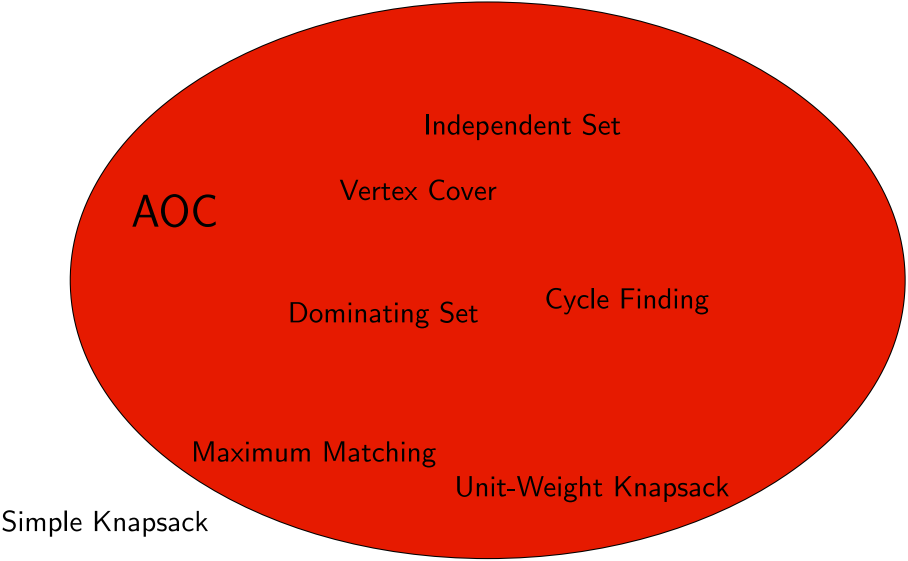
Example problems in AOC:
Independent Set, Disjoint Path Allocation.

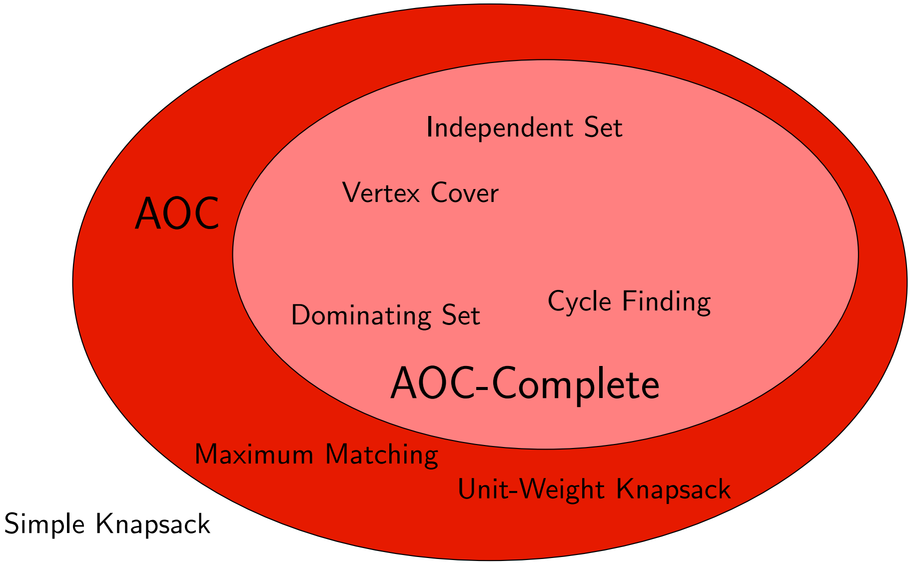
AOC-Complete

Definition

A problem in AOC is *AOC-Complete* if $\log\left(1+\frac{(c-1)^{c-1}}{c^c}\right)n - O(\log n)$ advice bits are necessary to achieve a competitive ratio of c.

Example AOC-Complete problems:
Vertex Cover, Dominating Set, Cycle Finding.


Independent Set
Vertex Cover
AOC
Dominating Set
Cycle Finding
Maximum Matching
Unit-Weight Knapsack
Simple Knapsack


Independent Set
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AOC-Complete
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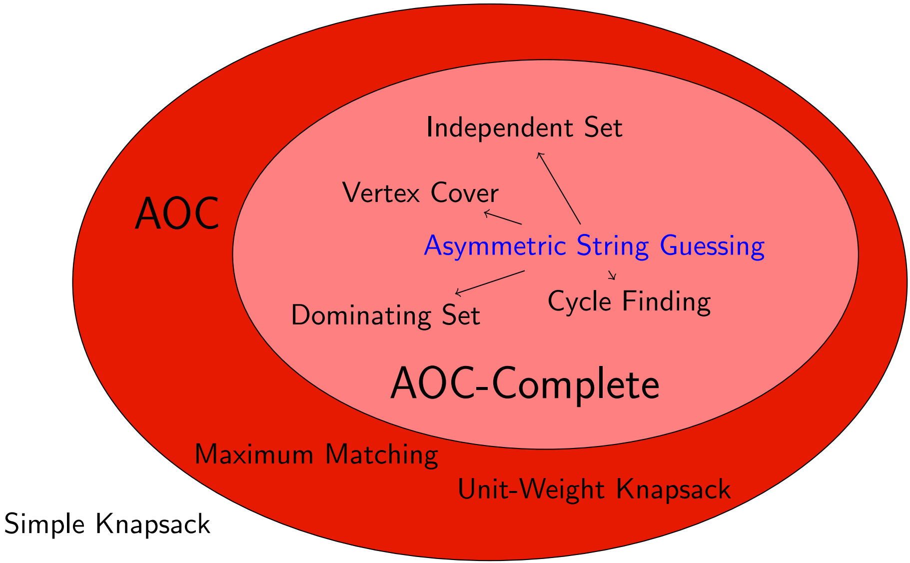
Independent Set
Vertex Cover
AOC
Asymmetric String Guessing
Cycle Finding
Dominating Set
AOC-Complete
Maximum Matching
Unit-Weight Knapsack
Simple Knapsack

Asymmetric string guessing

Similar to string guessing.

Asymmetric string guessing

Similar to string guessing.

Definition

The online problem, MINASG, is as follows:

- The input is a (secret) string $x \in \{0,1\}^n$.
- In round i, the algorithm answers $a_i \in \{0,1\}$.
- The correct answer, x_i is then revealed (known history).
- If $x_i = 1$ and $a_i = 0$, the algorithm loses (cost ∞).
- The cost of a feasible solution is $\sum_{i=1}^{n} a_i$.
- The goal is to minimize this cost.

Asymmetric string guessing

Results:

- MINASG is in AOC.
- MINASG is AOC-Complete:

$$\log\left(1+\frac{(c-1)^{c-1}}{c^c}\right)n - O(\log n)$$

  advice bits are necessary to achieve a competitive ratio of c.
- MINASG can be reduced to other problems.
  - Vertex Cover
  - Dominating Set
  - Set Cover
  - Cycle Finding

Asymmetric string guessing is in AOC

Recall:

Definition

An accept/reject minimization problem is in *(AOC)*, *Asymmetric Online Covering*, if, for an online solution, Y:

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Interpret guessing $a_i = 1$ as accept and $a_i = 0$ as reject.
A solution is the set of indices where $a_i = 1$.

Asymmetric string guessing is in AOC

Recall:

Definition

An accept/reject minimization problem is in *(AOC), Asymmetric Online Covering*, if, for an online solution, Y:

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Interpret guessing $a_i = 1$ as accept and $a_i = 0$ as reject.
A solution is the set of indices where $a_i = 1$.

If a set is feasible, $(x_i = 1) \Rightarrow (a_i = 1)$, so any superset is feasible.
Clearly, ASG is in AOC.

Asymmetric string guessing is AOC-Complete

Theorem

A c-competitive algorithm, ALG*, for* MIN*ASG must read at least b bits advice, where*

$$b \geq \log_2 \left(\max_{t:\lfloor ct \rfloor < n} \frac{\binom{n}{t}}{\binom{\lfloor ct \rfloor}{t}} \right) - O(\log_2 n) = \log \left(1 + \frac{(c-1)^{c-1}}{c^c} \right) n - O(\log n)$$

Pf sketch (by contradiction)

Let $b =$ max number of advice bits read, given n, c.

Suppose $\exists t$, s.t. $\lfloor ct \rfloor < n$, $b < \log_2 \left(\frac{\binom{n}{t}}{\binom{\lfloor ct \rfloor}{t}} \right)$.

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Suppose $\exists t$, s.t. $\lfloor ct \rfloor < n$, $b < \log_2 \left(\frac{\binom{n}{t}}{\binom{\lfloor ct \rfloor}{t}} \right)$.

$I_{n,t} = \{\bar{x} \in \{0,1\}^n \mid t = \sum_{i=1}^{n} x_i\}$
$|I_{n,t}| = \binom{n}{t}$
Let the set of strings where the Oracle gives advice ϕ be $I_{n,t}^{\phi} \subseteq I_{n,t}$.

ASG is AOC-Complete, cont.

Since $b < \log_2 \left(\frac{\binom{n}{t}}{\binom{\lfloor ct \rfloor}{t}} \right)$, number of different advice strings $< \frac{\binom{n}{t}}{\binom{\lfloor ct \rfloor}{t}}$.
By Pigeonhole Principle and $|I_{n,t}| = \binom{n}{t}$,

$$\exists \phi \;\; \text{s.t.} \;\; |I^{\phi}_{n,t}| > \binom{\lfloor ct \rfloor}{t}$$

ASG is AOC-Complete, cont.

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$$\exists \phi \ \text{ s.t. } \ |I^{\phi}_{n,t}| > \binom{\lfloor ct \rfloor}{t}$$

Claim $\exists \bar{x} \in I^{\phi}_{n,t}$ where $\textsc{Alg}$ answers 1 at least $\lfloor ct \rfloor + 1$ times.

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Claim $\exists \bar{x} \in I^{\phi}_{n,t}$ where $\textsc{Alg}$ answers 1 at least $\lfloor ct \rfloor + 1$ times.

Given $\textsc{Alg}$ and $I^{\phi}_{n,t}$, create an Adversary which forces this.

Consider strings in $I^{\phi}_{n,t}$ which are possible after Adversary has revealed some bits.

ASG is AOC-Complete, cont.

$$I_\phi = \begin{array}{c|c} 001 & {\color{red}0}01110 \\ 001 & {\color{red}0}01101 \\ 001 & {\color{red}0}01011 \\ 001 & {\color{red}0}00111 \\ 001 & {\color{red}0}10011 \end{array}$$

Note: $n = 9$, $t = 4$. $\langle x_1, x_2, x_3 \rangle = \langle 0, 0, 1 \rangle$ already known.

ASG is AOC-Complete, cont.

$$I_\phi = \begin{array}{c|c} 001 & 001110 \\ 001 & 001101 \\ 001 & 001011 \\ 001 & 000111 \\ 001 & 010011 \end{array}$$

Note: $n = 9$, $t = 4$. $\langle x_1, x_2, x_3 \rangle = \langle 0, 0, 1 \rangle$ already known.

ALG is guessing x_4. It can answer $a_4 = 0$.

ASG is AOC-Complete, cont.

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ALG is guessing x_4. It can answer $a_4 = 0$.

Adversary has to reveal $x_4 = 0$.

ASG is AOC-Complete, cont.

$$I_\phi = \begin{array}{c|c} 0010 & 01110 \\ 0010 & 01101 \\ 0010 & 01011 \\ 0010 & 00111 \\ 0010 & 10011 \end{array}$$

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Note: $n = 9$, $t = 4$. $\langle x_1, x_2, x_3, x_4 \rangle = \langle 0, 0, 1, 0 \rangle$ already known.

ALG is guessing x_5. It has to answer $a_5 = 1$.
Otherwise Adversary chooses last string 001010011, and ALG has lost.

ASG is AOC-Complete, cont.

$$I_\phi = \begin{array}{c|c} 0010 & 01110 \\ 0010 & 01101 \\ 0010 & 01011 \\ 0010 & 00111 \\ 0010 & 10011 \end{array}$$

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ALG is guessing x_5. It has to answer $a_5 = 1$.
Otherwise Adversary chooses last string 001010011, and ALG has lost.

Suppose Adversary reveals $x_5 = 1$.
Then the new $I_\phi = \{001010011\}$. ALG makes no more errors.

ASG is AOC-Complete, cont.

$$I_\phi = \begin{array}{c|c} 0010 & 01110 \\ 0010 & 01101 \\ 0010 & 01011 \\ 0010 & 00111 \\ 0010 & 10011 \end{array}$$

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Otherwise Adversary chooses last string 001010011, and ALG has lost.

Suppose Adversary reveals $x_5 = 0$.

ASG is AOC-Complete, cont.

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Note: $n = 9$, $t = 4$. $\langle x_1, x_2, x_3, x_4 \rangle = \langle 0, 0, 1, 0 \rangle$ already known.

ALG is guessing x_5. It has to answer $a_5 = 1$.
Otherwise Adversary chooses last string 001010011, and ALG has lost.

Suppose Adversary reveals $x_5 = 0$.
Then the new I_ϕ contains 4 strings. ALG must answer 1 for remaining bits.

ASG is AOC-Complete, cont.

Round i:

Let $m = |I_\phi|$, $h =$ number of 1's remaining in each string.
Let $m_0 =$ number of strings in I_ϕ where $x_i = 0$. $m_1 = m - m_0$.

ASG is AOC-Complete, cont.

Round i:

Let $m = |I_\phi|$, $h =$ number of 1's remaining in each string.
Let $m_0 =$ number of strings in I_ϕ where $x_i = 0$. $m_1 = m - m_0$.

If $m_0 = m$, Adversary answers $x_i = 0$.
If $m_0 < m$, look at minimum number of columns with ones:

- Let $d_1 = \min\{d' \mid m_1 \leq \binom{d'}{h-1}\}$
- Let $d = \min\{d' \mid m \leq \binom{d'}{h}\}$
- If $d_1 + 1 \geq d$ then

  Adversary answers $x_i = 1$.

  Otherwise

  Adversary answers $x_i = 0$.

ASG is AOC-Complete, cont.

$$I_\phi = \begin{array}{c|c} 0010 & 01110 \\ 0010 & 01101 \\ 0010 & 01011 \\ 0010 & 00111 \\ 0010 & 10011 \end{array}$$

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ALG is guessing x_5. It has to answer $a_5 = 1$.

Note: $m = 5$, $h = 3$, $m_0 = 4$, $m_1 = 1$.
$d_1 = \min\{d' \mid m_1 \leq \binom{d'}{h-1}\} = \min\{d' \mid 1 \leq \binom{d'}{2}\} = 2$
Let $d = \min\{d' \mid m \leq \binom{d'}{h}\} = \min\{d' \mid 5 \leq \binom{d'}{3}\} = 5$

These are the number of remaining columns where ALG is forced to answer $a_j = 1$ (for d_1, not counting the current column).

ASG is AOC-Complete, cont.

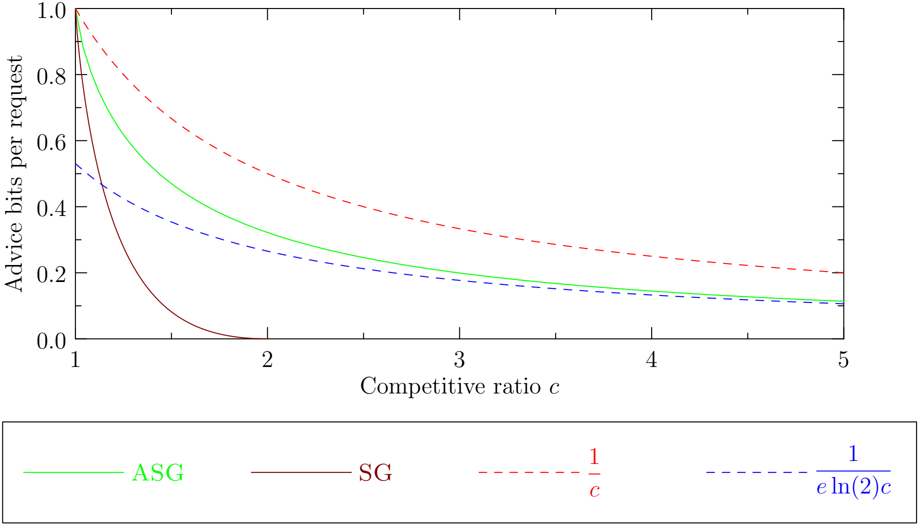
Using this adversary, the total number of columns (indices) where $\textsc{Alg}$ needs to answer 1 does not decrease.

ASG is AOC-Complete, cont.

Using this adversary, the total number of columns (indices) where $\textsc{Alg}$ needs to answer 1 does not decrease.

Let $L(m,h)$ denote the minimum cost Adversary can force.
$L(m,h) \geq \min\{d \mid m \leq \binom{d}{h}\}$.
Initially, $m > \binom{\lfloor ct \rfloor}{t}$ and $h = t$, so the minimum cost is at least $\lfloor ct \rfloor + 1$.
Contradiction

Advice bits needed to obtain competitive ratio c



Vertex Cover is AOC-Complete

Recall that Vertex Cover is in AOC.
To show completeness, we reduce from Asymmetric String Guessing:

$$x = \langle x_1, x_2, \ldots, x_n \rangle \longrightarrow \begin{array}{rcl} V & = & \{v_1, v_2, \ldots, v_n\}, \\ E & = & \{(v_i, v_j) \mid x_i = 1 \text{ and } i < j\}. \end{array}$$

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To show completeness, we reduce from Asymmetric String Guessing:

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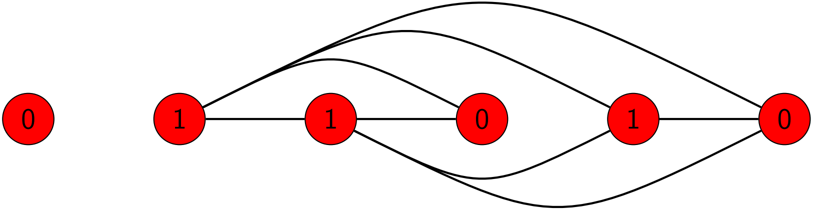
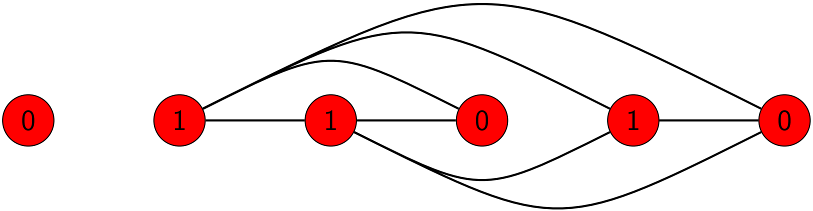


Vertex Cover is AOC-Complete, cont.

- Suppose we have a c-competitive algorithm, ALG, and an oracle, O, for Vertex Cover.
- Construct a c-competitive algorithm, ALG', and an oracle, O', for ASG.

$$x = \langle x_1, x_2, \ldots, x_n \rangle \longrightarrow \begin{array}{rcl} V & = & \{v_1, v_2, \ldots, v_n\}, \\ E & = & \{(v_i, v_j) \mid x_i = 1 \text{ and } i < j\}. \end{array}$$

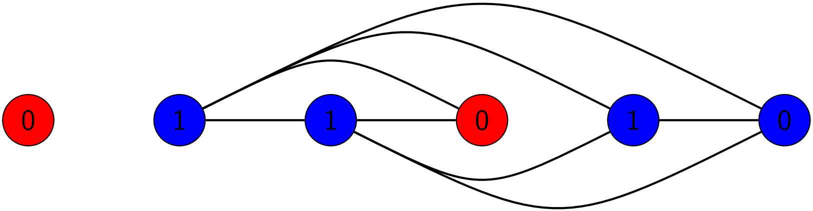


Vertex Cover is AOC-Complete, cont.

If ALG accepts all 1-vertices, ALG' answers 1 iff ALG accepts.

Since ALG is c-competitive, ALG' is too.

$$x = \langle x_1, x_2, \ldots, x_n \rangle \longrightarrow \begin{array}{rcl} V & = & \{v_1, v_2, \ldots, v_n\}, \\ E & = & \{(v_i, v_j) \mid x_i = 1 \text{ and } i < j\}. \end{array}$$



ALG' answers $\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle = \langle 0, 1, 1, 0, 1, 1 \rangle$.

Vertex Cover is AOC-Complete, cont.

If ALG rejects some 1-vertex, let v_i be the first it rejects.

Vertex Cover is AOC-Complete, cont.

If ALG rejects some 1-vertex, let v_i be the first it rejects.

Oracle, O', specifies index i and index j of some 0-vertex accepted.

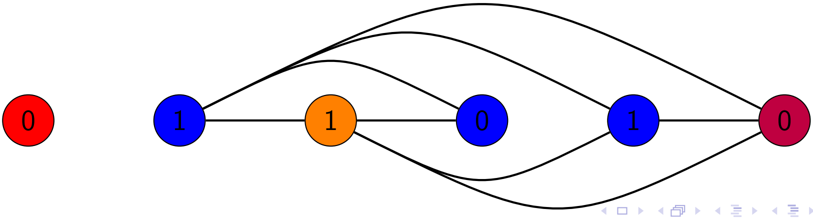
Vertex Cover is AOC-Complete, cont.

If ALG rejects some 1-vertex, let v_i be the first it rejects.

Oracle, O', specifies index i and index j of some 0-vertex accepted.

For index i, ALG' answers 1.
For index j, ALG' answers 0.
For all other indices, ALG' answers 1 iff ALG accepts.

ALG' answers 1 as many times as ALG accepts.
Since ALG is c-competitive, ALG' is too.



AOC-Complete problems

- Examples of AOC-Complete problems:
  - Vertex Cover
  - Dominating Set
  - Independent Set
  - Disjoint Path Allocation
  - Set Cover
- There are problems in AOC, which are not complete:
  - Unit-Weight Knapsack
  - Maximum Matching

Comparison with previous results

- Independent Set
  [Halldórsson,Iwama,Miyazaki,Taketomi, 2009]
  - Upper bound n/c
  - Lower bound $n/2c$
- Disjoint Path Allocation
  [Böckenhauer,Komm,Královič,Královič,Mömke, 2009]
  - Upper bound of $O\left(\frac{n \log_2 c}{c}\right)$
  - Lower bound of $n/2c$
- All other problems: No previous results
- All ASG and AOC results
  [B.,Favrholdt,Kudahl,Mikkelsen, 2017]

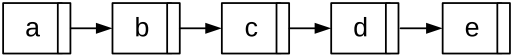
Section 4

Randomization and list access

List Accessing Problem

- Requests: for items in a list.
- Cost of accessing an item in index *i* is *i*.

$$< \mathbf{d\ b\ b\ d\ c\ a\ c} >$$



List Accessing Problem

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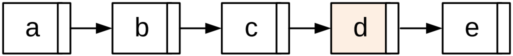
$$< \mathbf{d}\ \mathbf{b}\ \mathbf{b}\ \mathbf{d}\ \mathbf{c}\ \mathbf{a}\ \mathbf{c} >$$

cost: 4



List Accessing Problem

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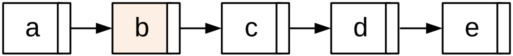
$$< d\ b\ b\ d\ c\ a\ c >$$

cost: $4 + 2$



List Accessing Problem

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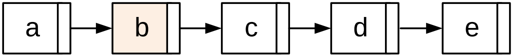
$< \mathbf{d\ b\ b\ d\ c\ a\ c} >$

cost: 4+2+2



List Accessing Problem

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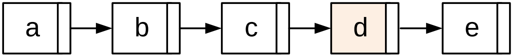
- Cost of accessing an item in index *i* is *i*.

< d b b d c a c >

cost: 4+2+2+4



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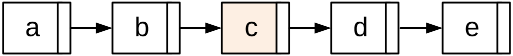
- Cost of accessing an item in index *i* is *i*.

$< \mathbf{d}\ \mathbf{b}\ \mathbf{b}\ \mathbf{d}\ \mathbf{c}\ \mathbf{a}\ \mathbf{c} >$

cost: 4+2+2+4+3

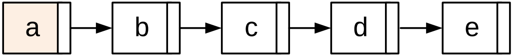


List Accessing Problem

- Requests: for items in a list.
- Cost of accessing an item in index *i* is *i*.

$$< \mathbf{d}\ \mathbf{b}\ \mathbf{b}\ \mathbf{d}\ \mathbf{c}\ \mathbf{a}\ \mathbf{c} >$$

cost: $4+2+2+4+3+1$

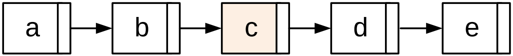


List Accessing Problem

- Requests: for items in a list.
- Cost of accessing an item in index i is i.

$$< \mathbf{d\ b\ b\ d\ c\ a\ c} >$$

$$\text{cost:} \quad 4{+}2{+}2{+}4{+}3{+}1{+}3 = 19$$



Self-Adjusting Lists

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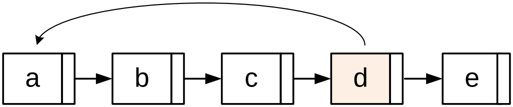
  - Paid exchanges: Swap positions of two consecutive items. Cost 1.

Move-To-Front (MTF)

- After each access, move the requested item to the front.
  - It only uses free exchanges.

$$< d\ b\ b\ d\ c\ a\ c >$$

cost: 4



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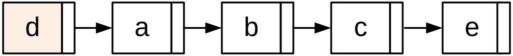
$$< \color{red}{d}\ b\ b\ d\ c\ a\ c >$$

cost: 4



Move-To-Front (MTF)

- After each access, move the requested item to the front.
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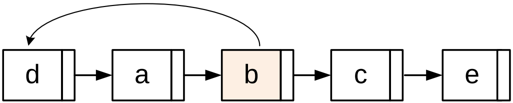
$<$ d **b** b d c a c $>$

cost: 4+3



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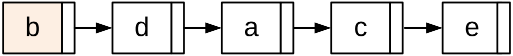

Move-To-Front (MTF)

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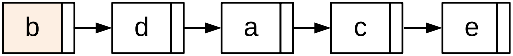
< d b b d c a c >

cost: 4+3+1



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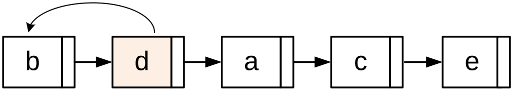
$$< d\ b\ b\ d\ c\ a\ c >$$

cost: 4+3+1+2



Move-To-Front (MTF)

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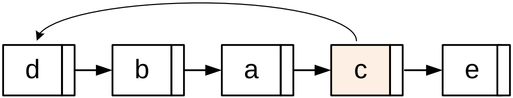
  - It only uses free exchanges.

$<$ d b b d c a c $>$

cost: 4+3+1+2+4



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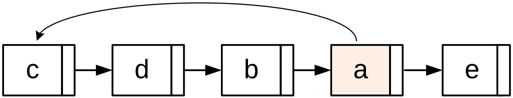

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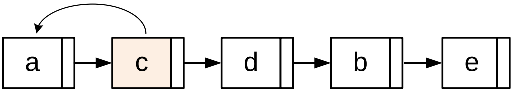

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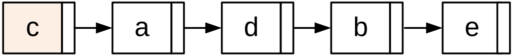
$$\langle\ d\ b\ b\ d\ c\ a\ c\ \rangle$$

cost: 4+3+1+2+4+4+2



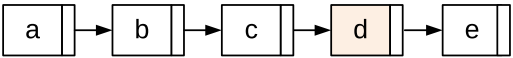
- Total cost: 20.

TIMESTAMP (TS)

- After an access to x,
  move x to the front of the first item y
  which has been requested at most once since the last access to x.
  - Do nothing if such an item y does not exist.
  - It only uses free exchanges.

$< d\ b\ b\ d\ c\ a\ c >$

cost: 4



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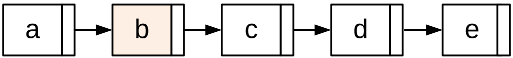
cost: 4+2



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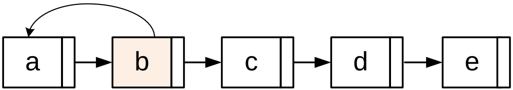

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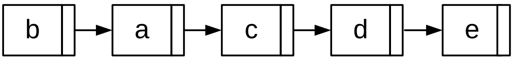

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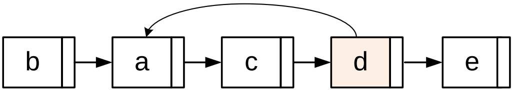
cost: 4+2+2+4



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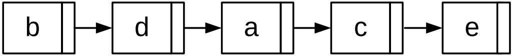

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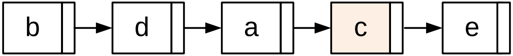

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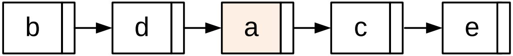
cost: 4+2+2+4+4+3



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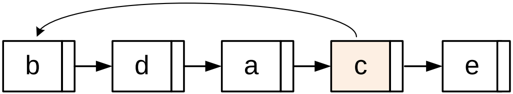

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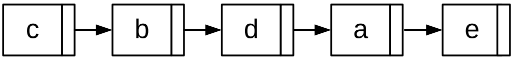

- Total cost: 23.

Competitive Analysis

- Competitive ratio of MTF for a list of length l is $2 - \frac{2}{l+1}$ [Sleator, Tarjan, 1985; Irani, 1991].
- Competitive ratio of TS is $2 - \frac{1}{l}$ [Albers, 1994].
- No deterministic online algorithm can have a competitive ratio better than $2 - \frac{2}{l+1}$ [Irani, 1991].

Breaking the lower bound

[B.,Kamali,Larsen,López-Ortiz, 2016]

- How many bits of advice are sufficient to perform strictly better than any online algorithm?
  - We show 2 bits of advice are sufficient.

Breaking the lower bound

[B.,Kamali,Larsen,López-Ortiz, 2016]

- How many bits of advice are sufficient to perform strictly better than any online algorithm?
  - We show 2 bits of advice are sufficient.
- We consider three classical algorithms and show that for any sequence, at least one of them has a performance ratio of at most $1.\bar{6}$:
  - TS (TIMESTAMP)
  - MTFE (Move-To-Front on Even accesses)
  - MTFO (Move-To-Front on Odd accesses)

MTF-every-other-access

MTFE and MTFO are the MTF-every-other-access (MTF2) algorithms.

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MTF2 *algorithms are 2.5-competitive.*

In an unpublished manuscript, 2003, Ansgar Grüne proves a lower bound of $\frac{7}{3}$ and claims an upper bound of 2.5.
We confirm that claim and raise the lower bound.

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MTF2 *algorithms are 2.5-competitive.*

In an unpublished manuscript, 2003, Ansgar Grüne proves a lower bound of $\frac{7}{3}$ and claims an upper bound of 2.5.
We confirm that claim and raise the lower bound.

Consider a list of ℓ items, initially ordered as $[a_1, a_2, \ldots, a_\ell]$.
Consider the following sequence of requests:

$$\sigma_m = \left\langle (a_1, a_2, \ldots, a_\ell, a_1^3, a_2^3, \ldots, a_\ell^3, a_\ell, a_{\ell-1}, \ldots, a_1, a_\ell^3, a_{\ell-1}^3, \ldots, a_1^3)^m \right\rangle .$$

We show that asymptotically, $\lim_{m\to\infty} \text{MTFO}(\sigma_m) = 2.5\text{OPT}(\sigma_m)$.

Breaking the lower bound

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- Phase-partitioning technique:
  - Compare the costs of the algorithms with $\textsc{Opt}$ for each projected sequence in *phases*.
  - Each phase ends with two consecutive requests to the same item.
  - We need to have the same phases for all algorithms.

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Type 1 phase: At start: $L = [x, y]$.
$\textsc{AlgMin}(\sigma) = \min\{\textsc{Mtfo}(\sigma), \textsc{Mtfe})(\sigma)\}$
$\textsc{AlgMax}(\sigma) = \max\{\textsc{Mtfo}(\sigma), \textsc{Mtfe})(\sigma)\}$
The costs are in the partial cost model.

Phase	$\textsc{AlgMin}$	$\textsc{AlgMax}$	TS	Sum ($\textsc{AlgMin}$ + $\textsc{AlgMax}$ + TS)	$\textsc{Opt}'$	$\frac{\text{Sum}}{\textsc{Opt}'}$
$x^j yy$	1	2	2	5	1	5
$x^j(yx)^{2i}yy$	$\leq 3i+1$	$\leq 3i+2$	$2 \times 2i = 4i$	$\leq 10i+3$	$2i+1$	< 5
$x^j(yx)^{2i-2}yxyy$	$\leq 3(i-1)+1$ $+\textsc{AlgMin}(\langle xyy\rangle)$	$\leq 3(i-1)+1$ $+\textsc{AlgMax}(\langle xyy\rangle)$	$2 \times (2i-1)$ $= 4i-2$	$\leq 6(i-1)+2+4$ $+(4i-2) = 10i-2$	$2i$	< 5
$x^j(yx)^{2i}x$	$\leq 3i$	$\leq 3i+1$	$2 \times 2i - 1$ $= 4i-1$	$\leq (6i+1)+(4i-1)$ $= 10i$	$2i$	≤ 5
$x^j(yx)^{2i-2}yxx$	$\leq 3(i-1)$ $+\textsc{AlgMin}(\langle yxx\rangle)$	$\leq 3(i-1)$ $+\textsc{AlgMax}(\langle yxx\rangle)$	$2 \times (2i-1)-1$ $= 4i-3$	$\leq 6(i-1)+4$ $+(4i-3) = 10i-5$	$2i-1$	≤ 5

Breaking the lower bound

Theorem

For any sequence σ *we have* $\mathrm{TS}(\sigma) + \textsc{Mtfo}(\sigma) + \textsc{Mtfe}(\sigma) \leq 5\,\textsc{Opt}(\sigma)$

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There is an algorithm that receives two bits of advice and achieves a competitive ratio of $1.\overline{6}$.

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There is an algorithm that receives two bits of advice and achieves a competitive ratio of $1.\overline{6}$.

- This can be regarded as the best existing (deterministic) approximation algorithm for the offline problem.
- If list access is used for file compression, adding two bits at the beginning of the file can guarantee better compression.

Randomization and advice

- The randomized algorithm which chooses each of TS, MTFE, and MTFO with probability $1/3$ is $1.\bar{6}$-competitive.
- If the number of algorithms was 2^k for some k,
  there would be a randomized algorithm using only k bits of randomness.
- A c-competitive randomized algorithm using $b(n)$ bits of randomness implies a c-competitive algorithm using at most $b(n)$ bits of advice.
- If there is provably no algorithm which is c-competitive using only $b(n)$ bits of advice, then there is no c-competitive randomized algorithm using only $b(n)$ bits of randomness.

Randomization and advice

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- [Böckenhauer,Komm,Královic,Královic, 2011]
  A c-competitive randomized algorithm
  implies a $(c+\epsilon)$-competitive algorithm using at most

$$b(n) = \lceil \log n \rceil + 2\lceil \log \lceil \log n \rceil \rceil + \log \left(\left\lfloor \frac{\log(m(n))}{\log(1+\epsilon)} \right\rfloor + 1 \right)$$

  bits of advice.

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  bits of advice.
  The number of different inputs of length n is $m(n)$.
  Holds for any $\epsilon > 0$.
- Now proving that any algorithm which is c-competitive requires enough advice, shows there is no c'-competitive randomized algorithm.

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The randomized k-server conjecture claims there is a randomized algorithm for the k-server problem which is $\Theta(\log k)$-competitive.

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[Böckenhauer,Komm,Královic,Královic, 2011]
If every online algorithm with advice for the k-server problem needs to use at least $\omega(\log n)$ advice bits to be $O(\log k)$-competitive, the randomized k-server conjecture does not hold.

Section 5

Concluding remarks

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  - Rule out possibilities for certain semi-online approaches.
  - Rule out possibilities for randomized approaches.

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  - Rule out possibilities for certain semi-online approaches.
  - Rule out possibilities for randomized approaches.
- Upper bounds can be either practical or purely theoretical.
- There is a start of a complexity theory, based on string guessing problems.

Thank you for your attention.