A Note on Lattices and Fixed Points

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March 2007

Abstract

We define complete lattices, and discuss the existence and construction of fixed points of monotone functions on these lattices. We also demonstrate how these results can be used for solving equational systems.

1 Introduction

In many areas of computer science, it is important to be able find fixed points for monotone functions or to be able to solve certain equational systems build from monotone functions.

There are numerous examples of applications of these techniques, and they come from many different areas such as compiler construction, type theory, database query optimization, deductive databases, logic programming, and semantics.

Sometimes, however, lattices do not provide the right tool for a specific application in which case complete partial orders can often be used instead. These structures have properties similar to complete lattices. We give a brief account of this in the concluding remarks.

2 Preliminaries

In this section, we define the necessary mathematical notions.

Definition 1 If U is a set and $\sqsubseteq$ a binary relation on U, then the system $(U, \sqsubseteq)$ is a *partial order* if:

- $\forall x \in U\colon\; x \sqsubseteq x$ (reflexitivity)
- $\forall x, y, z \in U\colon\; (x \sqsubseteq y \wedge y \sqsubseteq z) \;\Rightarrow\; x \sqsubseteq z$ (transitivity)
- $\forall x, y \in U\colon\; (x \sqsubseteq y \wedge y \sqsubseteq x) \;\Rightarrow\; x = y$ (antisymmetry)

$\square$

Example 1 Consider the set of integers $\mathcal{Z}$ with the usual ordering $\leq$. Define the set U by $U = \{(x, y) \mid x, y \in \mathcal{Z}\}$, and define $\sqsubseteq$ by $(x, y) \sqsubseteq (u, v)$ if $x \leq u$ and $y \leq v$. Then $(U, \sqsubseteq)$ is a partial order. $\square$

Definition 2 Assume that $(U, \sqsubseteq)$ is a partial order and assume that $A \subseteq U$. If there exists an element $z \in U$ such that

- $\forall x \in A\colon\; x \sqsubseteq z$
- $\forall y \in U\colon\; (\forall x \in A\colon\; x \sqsubseteq y) \;\Rightarrow\; z \sqsubseteq y$

then z is called the *least upper bound* of A. The least upper bound of A is denoted $\sqcup A$. $\square$

The first condition in Definition 2 states that $\sqcup A$ is in fact an upper bound, and the second that it is the least.

Usually, the least upper bound of a set containing exactly two elements is written infix, i.e., if $A = \{a, b\}$, then $\sqcup A$ can also be written $a \sqcup b$.

Proposition 1 The least upper bound is well-defined, i.e., at most one element can fulfill the two conditions from Definition 2. $\square$

Exercise 1 Prove proposition 1. $\square$

Similar to the least upper bound, we define the following:

Definition 3 Assume that $(U, \sqsubseteq)$ is a partial order, and assume that $A \subseteq U$. If there exists an element $z \in U$ such that

- $\forall x \in A\colon\; z \sqsubseteq x$
- $\forall y \in U\colon\; (\forall x \in A\colon\; y \sqsubseteq x) \;\Rightarrow\; y \sqsubseteq z$

then z is called the *greatest lower bound* of A. The greatest lower bound of A is denoted $\sqcap A$. $\square$

If A contains exactly two elements a and b, then $\sqcap A$ can also be written $a \sqcap b$.

Lattices, Functions, and Fixed Points

In this section, we define lattices and complete lattices. Then we consider monotone functions on these lattices, and we prove results about the fixed points of these functions.

Definition 4 If $(U, \sqsubseteq)$ is a partial order where $U \neq \emptyset$, and for all $x, y \in U$, $x \sqcup y$ and $x \sqcap y$ exist, then the system $(U, \sqsubseteq)$ is called a *lattice*.

If $\sqcup A$ and $\sqcap A$ exist for arbitrary subsets A of U, then the system $(U, \sqsubseteq)$ is called a *complete* lattice. $\square$

Two elements of a complete lattice $(U, \sqsubseteq)$ are particularly interesting: the element $\top = \sqcup U$ (top) and $\bot = \sqcap U$ (bottom). Clearly, $\forall x \in U\colon\ \bot \sqsubseteq x \sqsubseteq \top$.

Exercise 2 Consider the partial order defined in Example 1. Show that $(U, \sqsubseteq)$ is a lattice, but not a complete lattice. $\square$

Definition 5 Let $(U, \sqsubseteq)$ be a lattice and assume that $a, b \in U$, where $a \sqsubseteq b$. Let $[a, b]$ denote the set $\{x \in U \mid a \sqsubseteq x \wedge x \sqsubseteq b\}$. The system $([a, b], \sqsubseteq)$ is called an *interval lattice* of $(U, \sqsubseteq)$. $\square$

Lemma 1 If $(U, \sqsubseteq)$ is a lattice and $([a, b], \sqsubseteq)$ is an interval lattice of $(U, \sqsubseteq)$, then $([a, b], \sqsubseteq)$ is a lattice. If $(U, \sqsubseteq)$ is complete, then $([a, b], \sqsubseteq)$ is also complete. $\square$

Exercise 3 Prove Lemma 1. $\square$

Definition 6 Let $(U_1, \sqsubseteq_1)$ and $(U_2, \sqsubseteq_2)$ be partial orders. A function $f : U_1 \to U_2$ is called *increasing* if $\forall x, y \in U_1 :\ x \sqsubseteq_1 y \ \Rightarrow\ f(x) \sqsubseteq_2 f(y)$. Similarly, f is called *decreasing* if $\forall x, y \in U_1 :\ x \sqsubseteq_1 y \ \Rightarrow\ f(y) \sqsubseteq_2 f(x)$. $\square$

All results in the rest of this paper are stated for increasing functions. Dual results exist for decreasing functions, but we shall not mention them here.

Definition 7 If U is a set and $f :\ U \to U$ is a function, then $u \in U$ is called a *fixed point* of f if $f(u) = u$.

A fixed point $u \in U$ of f is called *minimal* if for all other elements $v \in U$, which are fixed points of f, we have that $v \not\sqsubseteq u$. If a function f has exactly one minimal fixed point, then this fixed point is called the *least* fixed point of f. $\square$

We can now prove that an increasing function on a complete lattice must have a least fixed point. In fact, we prove a more general statement.

The following theorem from [1] is often referred to as Tarski's theorem. Tarski himself called it the *lattice-theoretical fixpoint theorem*. The proof is a more detailed version of the proof of Tarski.

Theorem 1 Assume that $(U, \sqsubseteq)$ is a complete lattice and that $f\colon U \to U$ is an increasing function. Let P be the set of all fixed points of f. Then $P \neq \emptyset$ and $(P, \sqsubseteq)$ is a complete lattice. Additionally,

$$\sqcup P = \sqcup\{x \in U \mid f(x) \sqsupseteq x\} \in P$$

and

$$\sqcap P = \sqcap\{x \in U \mid f(x) \sqsubseteq x\} \in P.$$

Proof Let $u = \sqcup\{x \in U \mid f(x) \sqsupseteq x\}$. First, we show that u is a fixed point of f.

For any x with $f(x) \sqsupseteq x$, we must have that $x \sqsubseteq u$, since u is an upper bound. As f is increasing, $f(x) \sqsubseteq f(u)$, so by transitivity, $x \sqsubseteq f(u)$. This means that $f(u)$ is also an upper bound for the set $\{x \in U \mid f(x) \sqsupseteq x\}$. Since u is the *least* such upper bound, $u \sqsubseteq f(u)$.

Since f is increasing, $u \sqsubseteq f(u)$ implies that $f(u) \sqsubseteq f(f(u))$, so $f(u) \in \{x \in U \mid f(x) \sqsupseteq x\}$. As u is an upper bound for this set, $f(u) \sqsubseteq u$.

By antisymmetry, we can now conclude that $f(u) = u$, so u is a fixed point of f, i.e., $u \in P$.

We now show that $u = \sqcup P$. Clearly, since $P \subseteq \{x \in U \mid f(x) \sqsupseteq x\}$, u is also an upper bound for P. As $u \in P$, u must be the least upper bound for P.

A similar argument establishes that $\sqcap P = \sqcap\{x \in U \mid f(x) \sqsubseteq x\} \in P$.

It remains to be proven that $(P, \sqsubseteq)$ is a complete lattice. Let Y be any subset of P. We must prove that Y has a least upper bound.

For this purpose, consider the interval lattice $([\sqcup Y, \top], \sqsubseteq)$ which, by Lemma 1, is complete. We want to prove that $f'\colon\ [\sqcup Y, \top] \to [\sqcup Y, \top]$ defined by $f'(x) = f(x)$, for all $x \in [\sqcup Y, \top]$, is in fact a function. It is necessary to demonstrate that $\forall x \in [\sqcup Y, \top]\colon\ f(x) \in [\sqcup Y, \top]$.

Let $x \in Y$. Since $\sqcup Y$ is an upper bound, $x \sqsubseteq \sqcup Y$, so $f(x) \sqsubseteq f(\sqcup Y)$, as f is increasing. The set Y contains only fixed points, so $f(x) = x$ and therefore, $x \sqsubseteq f(\sqcup Y)$. This means that $f(\sqcup Y)$ is also an upper bound for Y. Since $\sqcup Y$ is the *least* upper bound, $\sqcup Y \sqsubseteq f(\sqcup Y)$.

Now, let $x \in [\sqcup Y, \top]$. By definition, $\sqcup Y \sqsubseteq x$, so $f(\sqcup Y) \sqsubseteq f(x)$, as f is increasing. By transitivity, $\sqcup Y \sqsubseteq f(x)$, so $f(x) \in [\sqcup Y, \top]$. We have proved that f' is well-defined.

In the above, we have already established that the greatest lower bound v of all fixed points of f' is itself a fixed point of f', and since $v \in [\sqcup Y, \top]$, we have that $\sqcup Y \sqsubseteq v$, so v is an upper bound for Y. Obviously, v is also a fixed point of f, so $v \in P$.

The element v must be the *least* fixed point of f which is an upper bound for all the elements in Y. This is seen as follows. Assume that w is an upper bound for Y which is a fixed point of f. Then $\sqcup Y \sqsubseteq w$, since $\sqcup Y$ is the *least* upper bound, so $w \in [\sqcup Y, \top]$, and thus, $f'(w)$ is defined and of course $f'(w) = w$. As v is the greatest lower bound for all such elements, $v \sqsubseteq w$. In other words, v is the least upper bound for Y in the system $(P, \sqsubseteq)$.

A similar argument demonstrates that any set $Y \subseteq P$ has a greatest lower bound. Thus, $(P, \sqsubseteq)$ is a complete lattice. $\square$

Knowing that there exists a least fixed point, it would be useful to have a simple method for computing it. It turns out that there is a simple iterative method which handles a frequently occurring special case, where U is finite.

Theorem 2 Let $(U, \sqsubseteq)$ be a complete lattice and $f\colon U \to U$ an increasing function. If U is finite, then the least fixed point of f can be found as $f^k(\bot)$ for some $k \in \mathbb{N}$.

Proof Let $u = \sqcup\{f^i(\bot) \mid i \in \mathbb{N}\}$. From $\bot \sqsubseteq f(\bot)$, it follows by simple induction (since f is increasing) that for all i, $f^i(\bot) \sqsubseteq f^{i+1}(\bot)$. So, we have for all i that $\sqcup\{\bot, f(\bot), \ldots, f^i(\bot)\} = f^i(\bot)$. If there are infinitely many i's such that $f^i(\bot) \sqsubset f^{i+1}(\bot)$, then U must be infinite. As this is not the case, there are only finitely many such i's. So, there exists a k such that for all $i \geq k$, we have that $f^i(\bot) = f^{i+1}(\bot)$. We conclude that $u = f^k(\bot)$.

First, we observe that u is indeed a fixed point as

$$u = f^k(\bot) = f(f^k(\bot)) = f(u).$$

Now assume that v is also a fixed point of f. We obtain that $f(\bot) \sqsubseteq f(v) = v$, as f is increasing. By simple induction, it follows that $f^k(\bot) \sqsubseteq v$. But then $u \sqsubseteq v$, so u is the least fixed point of f. $\square$

Proposition 2 Let $(U, \sqsubseteq)$ be a complete lattice and $f\colon\ U \rightarrow U$ an increasing function. If U is finite and $u \in U$ is smaller than the least fixed point of f, then the least fixed point of f can be found as $f^k(u)$ for some $k \in I\!N$. □

Exercise 4 Prove Proposition 2, which is a stronger version of Theorem 2. □

In the next section, we need product lattices. They are defined as follows:

Definition 8 Assume that $(U_1, \sqsubseteq_1), \ldots, (U_n, \sqsubseteq_n)$ are all lattices, Assume further that for all i, $(U_i, \sqsubseteq_i)$ has least upper bound and greatest lower bound operators $\sqcup_i$ and $\sqcap_i$, respectively. Then the *product lattice* $(U, \sqsubseteq)$ of

$$(U_1, \sqsubseteq_1), \ldots, (U_n, \sqsubseteq_n)$$

is defined by

- $U = U_1 \times \cdots \times U_n$
- $(x_1, \ldots, x_n) \sqsubseteq (y_1, \ldots, y_n) \;\Leftrightarrow\; (x_1 \sqsubseteq_1 y_1) \wedge \cdots \wedge (x_n \sqsubseteq_n y_n)$
- $(x_1, \ldots, x_n) \sqcup (y_1, \ldots, y_n) = (x_1 \sqcup_1 y_1, \ldots, x_n \sqcup_n y_n)$
- $(x_1, \ldots, x_n) \sqcap (y_1, \ldots, y_n) = (x_1 \sqcap_1 y_1, \ldots, x_n \sqcap_n y_n)$

□

Proposition 3 A product lattice $(U, \sqsubseteq)$ is a lattice. □

Exercise 5 Prove Proposition 3. □

Lemma 2 If $(U, \sqsubseteq)$ is a complete lattice, then the infix versions of $\sqcup$ and $\sqcap$, considered as binary operators, are commutative and associative, i.e., for all $x, y, z \in U$:

- $x \sqcup y = y \sqcup x$
- $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

and

- $x \sqcap y = y \sqcap x$ □
- $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$

Exercise 6 Prove Lemma 2. □

In the light of Lemma 2, if we know that $(U, \sqsubseteq)$ is complete, and we know how the binary version of $\sqcup$ is defined, then the least upper bound of a set $A = \{x_1, x_2 \ldots, x_n\}$ can meaningfully be defined by $\sqcup A = x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n$ (why?). Similarly for the greatest lower bound.

Proposition 4 If a product lattice is a product of all complete lattices, then it is itself complete. □

Exercise 7 Prove Proposition 4, defining the general least upper bound and greatest lower bound from the binary versions, as justified by Lemma 2. □

3 Equational Systems

In this section, we define equational systems over complete lattices, and we develop a method for finding the least solution to such systems.

Definition 9 An *equational system* over a complete product lattice $(U, \sqsubseteq)$, where $U = U_1 \times \cdots \times U_n$, consists of n equations

$$\begin{aligned} M_1 &= f_1(M_1, \ldots, M_n) \\ M_2 &= f_2(M_1, \ldots, M_n) \\ &\vdots \\ M_n &= f_n(M_1, \ldots, M_n) \end{aligned}$$

where the M_j's are variables (they are all different) and for all $i \in \{1, \ldots, n\}$, $f_i \colon U \to U_i$ is an increasing function.

A *solution* $\mathcal{L}$ to an equational system assigns to each variable M some value $\mathcal{L}(M)$ such that all the equalities hold. □

In the following, we let $\overline{x}$ denote $(x_1, \ldots, x_n)$.

Lemma 1 Let $\mathcal{S}$ be an equational system over the complete product lattice $(U, \sqsubseteq)$. Then there exists a unique least solution to $\mathcal{S}$. Furthermore, it can be found as the least fixed point of $F : U \to U$ defined by $F(\overline{x}) = (f_1(\overline{x}), \ldots, f_n(\overline{x}))$.

Proof Since $f_1, \ldots, f_n$ are increasing functions, so is F. Clearly, any solution to $\mathcal{S}$ must be a fixed point of F. Likewise, any fixed point of F must be a solution to $\mathcal{S}$. The result then follows from Theorem 1. $\square$

One application for the results above is of particular interest to us. That is when the set U is the power set of some base set B.

Proposition 5 If B is a finite set, then $(2^B, \subseteq)$ with least upper bound and greatest lower bound operators $\cup$ and $\cap$, respectively, is a complete lattice. $\square$

Exercise 8 Prove Proposition 5 (recall that the union and the intersection of arbitrary collections of sets are always defined). $\square$

In the following, we let $\bot_i$ denote the bottom element of the complete lattice $(U_i, \sqsubseteq_i)$.

Theorem 3 Let $\mathcal{S}$ be an equational system over the finite complete product lattice $(U, \sqsubseteq)$. Then there exists a unique least solution to $\mathcal{S}$. Furthermore, it can be found as $F^k(\bot_1, \ldots, \bot_n)$, for some $k \in I\!N$ and some function F.

Proof Using the right-hand side functions of the equations in $\mathcal{S}$, we define $F\colon\ U \to U$ by $F(\overline{x}) = (f_1(\overline{x}), \ldots, f_n(\overline{x}))$. From Lemma 1, it follows that the least fixed point of F is the least solution to $\mathcal{S}$. As $(\bot_1, \ldots, \bot_n)$ is clearly the bottom element of U, it follows from Theorem 2 that there exists a $k \in I\!N$ such that this fixed point equals $F^k(\bot_1, \ldots, \bot_n)$. $\square$

Corollary 1 The least solution to equational systems over finite product lattices with n set-valued variables can be solved by iterating from the n-tuple $(\emptyset, \ldots, \emptyset)$.

Proof Directly from Theorem 3 and Proposition 5. $\square$

Exercise 9 Now we want the possibility of having more than one equation per variable, i.e., we do no longer wish to require that all the left-hand side variables (the M_j's) of an equational system be different. How can we then solve the system?

Hint: The operator $\sqcup$ can be viewed as an increasing function, and the set of increasing functions is closed under function composition. $\square$

Exercise 10 An *inequational system* is like an equational system, except that the equality signs ($=$) are replaced with inequality signs ($\sqsubseteq$). Given an inequational system, the equational system created by replacing all inequality signs with equality signs is called the *corresponding equality system*.

Show that an inequational system has a solution if and only if the corresponding equality system has a solution.

Show that if an inequational system and the corresponding equational system both have a solution, then their least solutions are identical. □

4 Concluding Remarks

For applications in Computer Science, we are usually also interested in knowing the complexity of a particular fixed point derivation. Nothing very specific can be said here, but we notice that when computing solutions by iterating a function from the bottom element, there are three costs to consider:

- the number of iterations to find the fixed point
- the cost of computing each element of the iteration
- the cost of checking whether or not a fixed point has been reached

Another issue is when the results presented in this note can be applied. Occasionally, applications are such that a finite and complete lattice cannot be established. In those cases, one might consider a related result described briefly below. The idea is to require less of the structure, but then more of the functions.

Assume that $(U, \sqsubseteq)$ is a partial order. A subset A of U is called a *chain* (or sometimes a *directed set*) if $\forall x, y \in A\colon\ x \sqsubseteq y \vee y \sqsubseteq x$.

The system $(U, \sqsubseteq)$ is called a *complete partial order* (or sometimes a *chain complete partial order*) if $\sqcup A$ is defined for all chains A.

Assume that $(U_1, \sqsubseteq_1)$ and $(U_2, \sqsubseteq_2)$ are complete partial orders with least upper bound operators $\sqcup_1$ and $\sqcup_2$, respectively. An increasing function $f\colon\ U_1 \rightarrow U_2$ is *continuous* if for all nonempty chains A, we have that $f(\sqcup_1 A) = \sqcup_2\{f(x) \mid x \in A\}$. Notice that the image of a chain is again a chain, so that the least upper bound is defined.

If f is a continuous function on a complete partial order $(U, \sqsubseteq)$, then the least fixed point of f exists, and it equals $\sqcup\{f_i(\perp) \mid i \geq 0\}$. Notice that

$\perp = \sqcup\emptyset$. Again, if U is not finite, then the least fixed point might not belong to $\{f^i(\perp) \mid i \geq 0\}$, so we cannot necessarily compute it by iteration.

References

[1] Alfred Tarski. A Lattice-Theoretical Fixpoint Theorem and its Applications. *Pacific J. Math*, 5:285–309, 1955.