Online Max-Edge-Coloring of Paths and Trees

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Edge Coloring

An edge coloring of the Petersen graph using 4 colors.
Minimum Edge Coloring

Classical edge coloring:

- Color the edges of a graph using as few colors as possible.
Minimum Edge Coloring

Classical edge coloring:
- Color the edges of a graph using as few colors as possible.

Vizing’s Theorem
Let $G$ be a simple graph of maximum degree $\Delta(G)$. The minimum number of colors needed to color all edges of $G$ is either $\Delta(G)$ or $\Delta(G) + 1$. 
Dual Edge Coloring

There is a dual version known as Edge-$k$-Coloring:
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- The goal is to color as many edges as possible.
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There is a dual version known as Edge-$k$-Coloring:

- A fixed number, $k$, of colors is available.
- The goal is to color as many edges as possible.

We label the $k$ colors $1, 2, \ldots, k$.
For $k = 1$, this is the maximum matching problem.
Online Edge Coloring

Online Edge-$k$-Coloring

- Edges arrive one by one.
- Must immediately color a newly arrived edge with one of the $k$ colors or reject the edge.
- The decision is *irrevocable*.
Example for $k = 2$
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Online Edge Coloring

Competitive analysis [Sleator, Tarjan ’85], [Karlin et al. ’88]

An algorithm A is \( c \)-competitive if

\[
A(\sigma) \geq c \cdot \text{OPT}(\sigma) - b
\]

for all sequence of edges \( \sigma \).

For a randomized algorithm, replace \( A(\sigma) \) with \( E[A(\sigma)] \).

The competitive ratio \( C = \sup\{c : A \text{ is } c\text{-competitive}\} \).
An algorithm $A$ is $c$-competitive if

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for all sequence of edges $\sigma$. For a randomized algorithm, replace $A(\sigma)$ with $E[A(\sigma)]$. The competitive ratio $C = \sup\{c : A \text{ is } c\text{-competitive}\}$. 

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (2,0);
\draw[fill=red] (1,0) circle (0.5cm);
\draw[fill=green] (0.5,0) circle (0.5cm);
\draw (0,0) node[below] {0} -- (2,0) node[below] {1};
\end{tikzpicture}
\end{center}
Previous results

(Favrholdt, Nielsen ’03)

- Negative results:

  No deterministic algorithm has a competitive ratio better than \( \frac{1}{2} \).
  No randomized algorithm has a competitive ratio better than \( \frac{4}{7} \).
Previous results

(Favrholdt, Nielsen ’03)

► Negative results:
No deterministic algorithm has a competitive ratio better than $\frac{1}{2}$.
No randomized algorithm has a competitive ratio better than $\frac{4}{7}$.

► Positive results:
The competitive ratio of a fair algorithm is at least $2\sqrt{3} - 3 \approx 0.46$
An algorithm is fair if it never rejects an edge unless forced to do so.
In order to obtain a more fine-grained analysis, we study Edge-$k$-Coloring on some basic graph classes:

- For paths, we give an optimal (randomized) algorithm.
- For trees, we show that a natural algorithm called First-Fit is optimal among deterministic algorithms.
- For trees and “tree-like” graphs, we show that any fair algorithm for online Edge-$k$-Coloring performs well if $k$ (the number of colors) is sufficiently large.
Why?

Why paths and trees?
Why paths and trees?

- Natural building blocks for studying more complicated graph classes.
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- Natural building blocks for studying more complicated graph classes.
- All previous (negative) results for Edge-$k$-Coloring holds when the input graph is a bipartite graph.
Recall that the competitive ratio of a fair algorithm is at least $2\sqrt{3} - 3 \approx 0.46$ and at most $\frac{1}{2}$ (Favrholdt, Nielsen ’03).

The following fair and deterministic algorithms have been studied:

- First-Fit uses the lowest available color when coloring an edge. It can be viewed as the natural greedy strategy.
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- **First-Fit** uses the lowest available color when coloring an edge. It can be viewed as the natural greedy strategy.

- **Next-Fit** remembers the last used color $c_{\text{last}}$. When coloring an edge, it uses the first available color in the ordered sequence $\langle c_{\text{last}} + 1, \ldots, k, 1, \ldots, c_{\text{last}} \rangle$. 
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- Next-Fit remembers the last used color $c_{last}$. When coloring an edge, it uses the first available color in the ordered sequence $\langle c_{last} + 1, \ldots, k, 1, \ldots, c_{last} \rangle$.

Next-Fit is shown to have a competitive ratio of exactly $2\sqrt{3} - 3$. The competitive ratio of First-Fit is shown to be at most 0.48.
Relationship to vertex coloring

- Edge coloring a graph $G$ is equivalent to vertex coloring the line graph of $G$.
- This also holds in an online setting.
- In particular, online Edge-$k$-Coloring on paths is exactly the same as online *dual* vertex coloring on paths.
Next-Fit has a competitive ratio of $\frac{1}{2}$ on paths.
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Edge-2-Coloring on Paths

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- First-Fit has a competitive ratio of $\frac{2}{3}$ on paths.
Edge-2-Coloring on Paths

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![Diagram of edge-2-coloring on a path]

Jesper W. Mikkelsen
Edge-2-Coloring on Paths

- Next-Fit has a competitive ratio of $\frac{1}{2}$ on paths.
- First-Fit has a competitive ratio of $\frac{2}{3}$ on paths.

No deterministic algorithm can do better than $\frac{2}{3}$. 
Edge-2-Coloring on Paths

- Can a randomized algorithm do better than $\frac{2}{3}$?
Edge-2-Coloring on Paths

- Can a randomized algorithm do better than $\frac{2}{3}$?
- Yes! There is a randomized algorithm with a competitive ratio of $\frac{4}{5}$. 
Let $\frac{1}{2} \leq p \leq 1$. Define $\operatorname{Rand}_p$ as follows:

- For isolated edges, use the color 1 with probability $p$ and the color 2 with probability $1 - p$. Non-isolated edges are colored if possible.
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- Two types of rejections:

\[
\begin{array}{c}
p \quad ? \quad p \\
\hline
\end{array}
\]

Dashed edge is colored with probability $p^2 + (1 - p)^2$. 
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- Two types of rejections:

  \[ p \quad ? \quad (1 - p) \quad p \]

  Dashed edge is colored with probability $p(1 - p) + (1 - p)p$. 
Choose the parameter $p$ so that we balance the two situations:

$p = \frac{\sqrt{5}}{\approx 0.72}$ gives a competitive ratio of $\frac{4}{5}$.
Randomization

- Can a randomized algorithm do better than $\frac{4}{5}$?
Randomization

- Can a randomized algorithm do better than $\frac{4}{5}$?
- No. We prove this using Yao’s minimax principle.
Edge-$k$-Coloring on Trees

- Suppose that the input graph is a tree.
Edge-$k$-Coloring on Trees

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- For $k \geq 2$, we show that:
  - The competitive ratio of any fair algorithm is at least $\frac{2 \sqrt{k-2}}{2 \sqrt{k-1}}$. 

First-Fit is optimal among deterministic or fair algorithms.
Suppose that the input graph is a tree.

For $k \geq 2$, we show that:

- The competitive ratio of any fair algorithm is at least $\frac{2\sqrt{k} - 2}{2\sqrt{k} - 1}$.
- The competitive ratio of First-Fit is exactly $\frac{k - 1}{k}$.
Suppose that the input graph is a tree.

For $k \geq 2$, we show that:

- The competitive ratio of any fair algorithm is at least $\frac{2\sqrt{k-2}}{2\sqrt{k-1}}$.
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- First-Fit is optimal among deterministic or fair algorithms.
First-Fit vs Next-Fit on Trees

![Graph showing the comparison between First-Fit (FF) and Next-Fit (NF) methods on trees. The y-axis represents the completion ratio, and the x-axis represents the value of k. The graph illustrates the performance of both methods as k increases.](image)
Charging technique for proving positive results

Three types of edges for a deterministic algorithm A:

- Double colored: Colored by both A and OPT.
- Single colored: Colored only by A.
- Rejected: Colored only by OPT.
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We want to prove that A is $C$-competitive. Suppose that A earns a dollar whenever it colors an edge. We need to show that A can buy all of the edges colored by OPT, paying at least $C$ for each.
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Single colored edges will have a surplus of 1.
Any fair algorithm $F$ has a competitive ratio of at least $C = \frac{2\sqrt{k} - 2}{2\sqrt{k} - 1}$.
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- Double colored edges have a surplus of $1 - C = \frac{1}{2\sqrt{k-1}}$.
- Single colored edges have a surplus of 1.
- Rejected edges need to receive a value of at least $C$ from the colored edges.
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- **Single** colored edges have a surplus of 1.
- **Rejected** edges need to receive a value of at least $C$ from the colored edges.
Strategy for redistributing the surplus:

\[ C = \frac{2\sqrt{k} - 2}{2\sqrt{k} - 1} \]

remaining surplus
What if a rejected edge $e$ has only a few colored child edges?
Fair Algorithm on Trees

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Worst-case: Roughly $\sqrt{k}$ colored child edges.
First-Fit on Trees

First-Fit has a competitive ratio of at least $\frac{k-1}{k}$ on trees. Use the same strategy as before with the following addition:
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Example: \( k = 5 \), only double colored.
First-Fit on Trees

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Example: $k = 5$, only double colored.

$m(v) = \frac{8}{5}$.

$v$ transfers $\frac{4}{5}$ to $(w,v)$ and $\frac{2}{5}$ to each of $(v,x)$ and $(v,y)$. 
First-Fit on Trees

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First-Fit has a competitive ratio of at least $\frac{k-1}{k}$ on trees.

**Step 1** Consider in turn all edges $e = (v, u) \in E_c$. Let $c$ be the color assigned to $e$ by First-Fit and let $e' = (w, v)$ be the parent edge of $e$.

- **Step 1.1** If $e' \in E_d$ and $e'$ has been colored with a color $c' > c$, then $e$ transfers a value of $\frac{1}{k}$ to $w$.
- **Step 1.2** Any surplus remaining at $e$ is transferred to $v$.

For each vertex $v$, let $m(v)$ denote the value transferred to $v$ in step 1.

**Step 2** Consider in turn all vertices $v \in V$.

- **Step 2.1** If $v$ has a parent edge $e'$ and $e' \in E_r$, then $v$ transfers a value of $\min \{m(v), \frac{k-1}{k}\}$ to $e'$.
- **Step 2.2** Any value remaining at $v$ is distributed equally among the child edges of $v$ belonging to $E_r$. 
Negative result for Edge-$k$-Coloring of Trees

First-Fit has a competitive ratio of at most $\frac{k-1}{k}$.

![Diagram of a tree with edges colored to illustrate the competitive ratio](image)
Negative result for Edge-$k$-Coloring of Trees

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First-Fit has a competitive ratio of at most \( \frac{k-1}{k} \).

\[
\begin{array}{c}
\text{1} \\
k \\
\text{2} \\
k \\
\text{1} \\
k
\end{array}
\]

First-Fit colors \( N(k-2) + N = N(k-1) \) and OPT colors \( Nk \) edges.
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A similar construction shows that no fair or deterministic algorithm can do better than $\frac{k-1}{k}$.
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Furthermore, one can show that the competitive ratio of Next-Fit is no better than $\frac{2\sqrt{k-2}}{2\sqrt{k-1}}$ when $k$ is a square number.
Randomization on Trees

- First-Fit is optimal on trees among fair or deterministic algorithms with a competitive ratio of $\frac{k-1}{k}$.
- Can a randomized algorithm do better than $\frac{k-1}{k}$?
Randomization on Trees

- First-Fit is optimal on trees among fair or deterministic algorithms with a competitive ratio of \( \frac{k-1}{k} \).
- Can a randomized algorithm do better than \( \frac{k-1}{k} \)?
- Maybe, but not better than \( \frac{k}{k+1} \).
If it looks like a tree...

- There exists several measures of how “tree-like” a graph is.
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- The *pseudoarboricity (PA)* of $G$ is the minimum $t$ such that the edges of $G$ can be oriented to form a digraph where each vertex has outdegree at most $t$. 

Trees have $PA = 1$. Planar graphs have $PA$ at most 3.

Graphs of bounded degree, treewidth, degeneracy or genus has bounded $PA$. 
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Parameterized Competitive Ratio

- We *parameterize* the competitive ratio by the PA of the input graph.
Parameterized Competitive Ratio

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**Theorem**
Suppose that the input graph is $k$-colorable and has PA at most $t$. If $t \leq \frac{1}{4}k$, then the competitive ratio of any fair algorithm is at least

$$\frac{2\sqrt{k/t} - 2}{2\sqrt{k/t} - 1}.$$
Parameterized Competitive Ratio

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The competitive ratio on $k$-colorable graph is also known as the competitive ratio on *accommodating sequences* [Boyar, Larsen, Nielsen ’98].
Parameterized Competitive Ratio

A lower bound for any fair algorithm on planar graphs (PA \( \leq 3 \)).
Conclusion

- $\text{Rand}_p$ is optimal on paths and better than any deterministic algorithm.
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- First-Fit is optimal among deterministic algorithms on paths and trees.
Conclusion

- $\text{Rand}_p$ is optimal on paths and better than any deterministic algorithm.
- First-Fit is optimal among deterministic algorithms on paths and trees.
- On tree-like graphs, any fair algorithm for online Edge-$k$-Coloring performs well if it has a sufficiently large number of colors.
Open Problems

- Find the optimal online algorithm for Edge-\(k\)-Coloring in general and on other graph classes.
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Does First-Fit have a competitive ratio better than $2\sqrt{3} - 3$ for Edge-$k$-Coloring?
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Is it possible to achieve a competitive ratio better than $2\sqrt{3} - 3$ for Edge-$k$-Coloring?
Does First-Fit have a competitive ratio better than $2\sqrt{3} - 3$ for Edge-$k$-Coloring? On bipartite graphs?
THANK YOU