Online Max-Edge-Coloring of Paths and Trees

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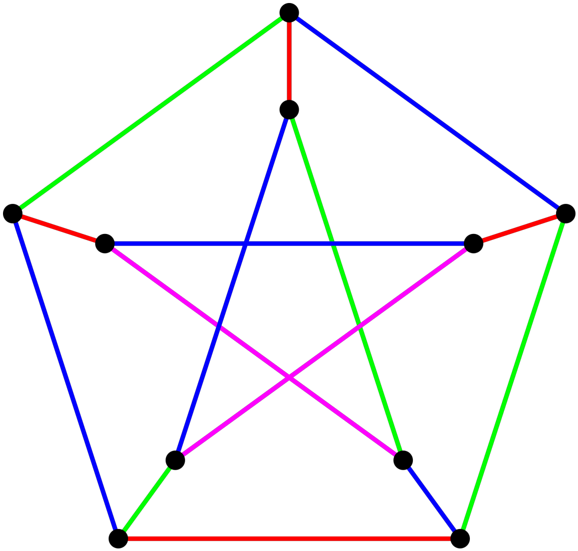
Edge Coloring



An edge coloring of the Petersen graph using 4 colors.

Minimum Edge Coloring

Classical edge coloring:

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Vizing's Theorem

Let G be a simple graph of maximum degree $\Delta(G)$. The minimum number of colors needed to color all edges of G is either $\Delta(G)$ or $\Delta(G) + 1$.

Dual Edge Coloring

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- A fixed number, k, of colors is available.
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We label the k colors $1, 2, \ldots, k$.
For $k = 1$, this is the maximum matching problem.

Online Edge Coloring

Online Edge-k-Coloring

- Edges arrive one by one.
- Must immediately color a newly arrived edge with one of the k colors or reject the edge.
- The decision is *irrevocable*.

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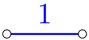

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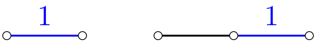

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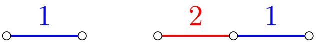

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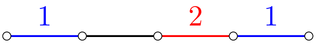

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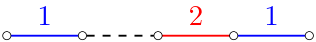

Online Edge Coloring

Competitive analysis [Sleator, Tarjan '85], [Karlin et al. '88]

An algorithm A is c-*competitive* if

$$\mathsf{A}(\sigma) \geq c \cdot \mathsf{OPT}(\sigma) - b$$

for all sequence of edges σ.
For a randomized algorithm, replace $\mathsf{A}(\sigma)$ with $E[\mathsf{A}(\sigma)]$.
The competitive ratio $C = \sup\{c : \mathsf{A} \text{ is } c\text{-competitive}\}$.

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(Favrholdt, Nielsen '03)

- Negative results:

No deterministic algorithm has a competitive ratio better than $\frac{1}{2}$.
No randomized algorithm has a competitive ratio better than $\frac{4}{7}$.

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- Positive results:

The competitive ratio of a *fair* algorithm is at least $2\sqrt{3} - 3 \approx 0.46$
An algorithm is *fair* if it never rejects an edge unless forced to do so.

What?

- In order to obtain a more fine-grained analysis, we study Edge-k-Coloring on some basic graph classes:
- For paths, we give an optimal (randomized) algorithm.
- For trees, we show that a natural algorithm called First-Fit is optimal among deterministic algorithms.
- For trees and "tree-like" graphs, we show that any fair algorithm for online Edge-k-Coloring performs well if k (the number of colors) is sufficiently large.

Why?

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- Natural building blocks for studying more complicated graph classes.
- All previous (negative) results for Edge-k-Coloring holds when the input graph is a bipartite graph.

Algorithms

Recall that the competitive ratio of a fair algorithm is at least $2\sqrt{3} - 3 \approx 0.46$ and at most $\frac{1}{2}$ (Favrholdt, Nielsen '03).
The following fair and deterministic algorithms have been studied:

- First-Fit uses the lowest available color when coloring an edge. It can be viewed as the natural greedy strategy.

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- Next-Fit remembers the last used color c_{last}. When coloring an edge, it uses the first available color in the ordered sequence $\langle c_{\text{last}} + 1, \ldots, k, 1, \ldots, c_{\text{last}} \rangle$.

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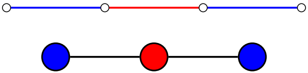
- Next-Fit remembers the last used color c_{last}. When coloring an edge, it uses the first available color in the ordered sequence $\langle c_{\text{last}} + 1, \ldots, k, 1, \ldots, c_{\text{last}} \rangle$.

Next-Fit is shown to have a competitive ratio of exactly $2\sqrt{3} - 3$.
The competitive ratio of First-Fit is shown to be at most 0.48.

Relationship to vertex coloring

- Edge coloring a graph G is equivalent to vertex coloring the line graph of G.
- This also holds in an online setting.
- In particular, online Edge-k-Coloring on paths is exactly the same as online *dual* vertex coloring on paths.



Edge-2-Coloring on Paths

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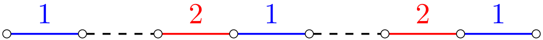

No deterministic algorithm can do better than $\frac{2}{3}$.

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Edge-2-Coloring on Paths

- Can a randomized algorithm do better than $\frac{2}{3}$?
- Yes! There is a randomized algorithm with a competitive ratio of $\frac{4}{5}$.

Rand$_p$

Let $\frac{1}{2} \leq p \leq 1$. Define Rand$_p$ as follows:

- For isolated edges, use the color 1 with probability p and the color 2 with probability $1 - p$. Non-isolated edges are colored if possible.

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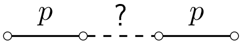

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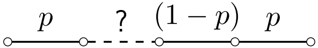
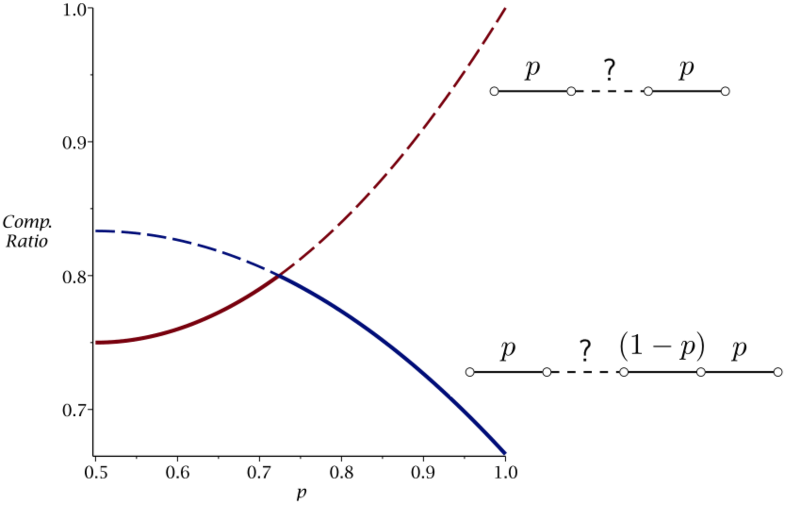


Dashed edge is colored with probability $p(1-p)+(1-p)p$.

Rand$_p$

Choose the parameter p so that we balance the two situations:
$p = \frac{\varphi}{\sqrt{5}} \approx 0.72$ gives a competitive ratio of $\frac{4}{5}$.



Randomization

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- No. We prove this using Yao's minimax principle.

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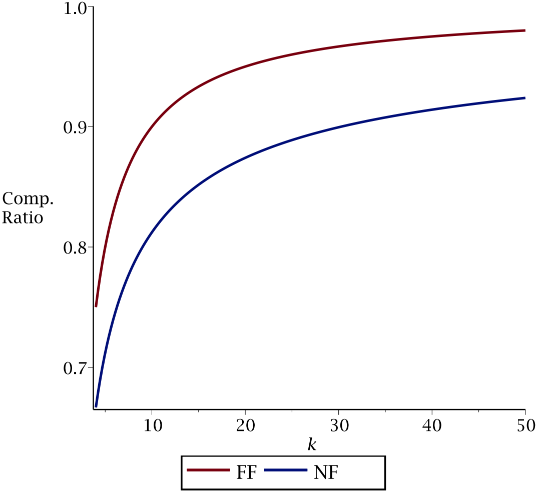
  - First-Fit is *optimal* among deterministic or fair algorithms.

First-Fit vs Next-Fit on Trees



Charging technique for proving positive results

Three types of edges for a deterministic algorithm A:

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Double colored edges will pay for themselves and therefore have a surplus of $1 - C$.
Single colored edges will have a surplus of 1.

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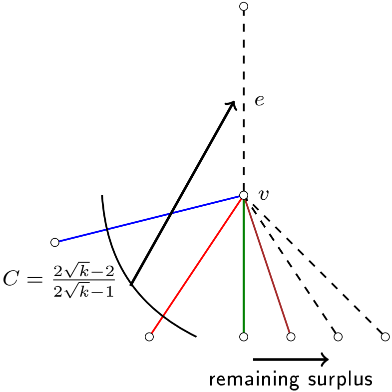
Fair Algorithm on Trees

Strategy for redistributing the surplus:



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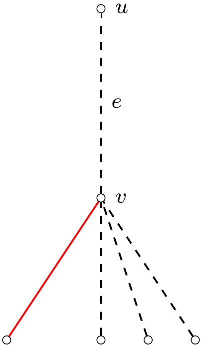

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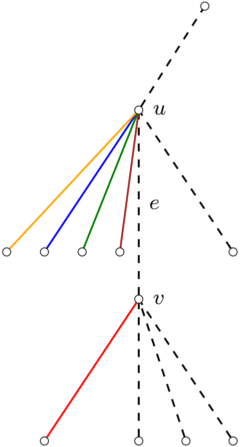

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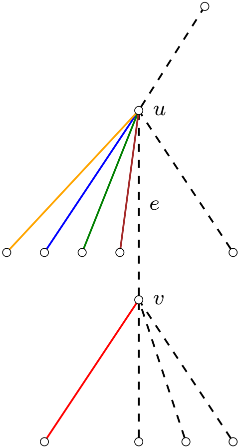

Worst-case: Roughly $\sqrt{k}$ colored child edges.

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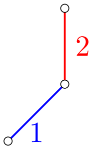

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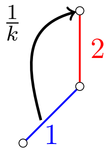
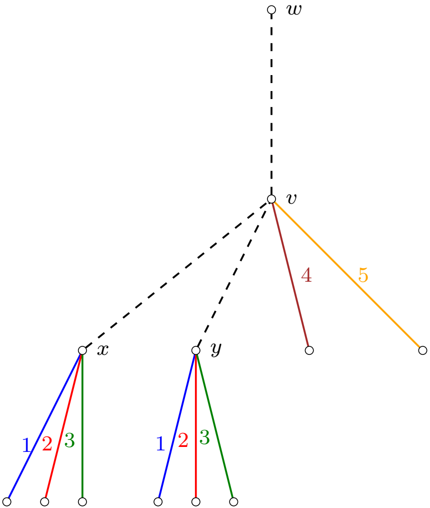


First-Fit on Trees

Example: $k = 5$, only double colored.



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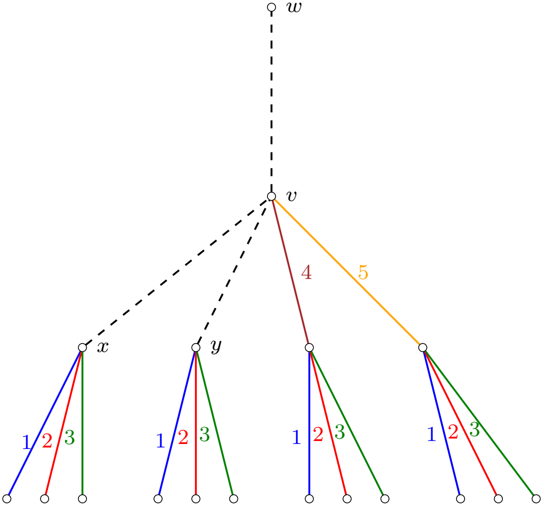

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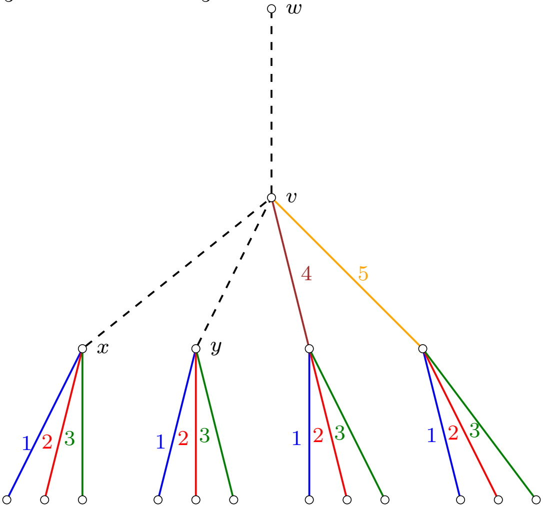
Example: $k = 5$, only double colored.
$m(v) = \frac{8}{5}$.
v transfers $\frac{4}{5}$ to (w, v) and $\frac{2}{5}$ to each of (v, x) and (v, y).



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Step 1 Consider in turn all edges $e = (v, u) \in E_{\mathsf{c}}$. Let c be the color assigned to e by First-Fit and let $e' = (w, v)$ be the parent edge of e.

- **Step 1.1** If $e' \in E_{\mathsf{d}}$ and e' has been colored with a color $c' > c$, then e transfers a value of $\frac{1}{k}$ to w.
- **Step 1.2** Any surplus remaining at e is transferred to v.

For each vertex v, let $m(v)$ denote the value transferred to v in step 1.

Step 2 Consider in turn all vertices $v \in V$.

- **Step 2.1** If v has a parent edge e' and $e' \in E_{\mathsf{r}}$, then v transfers a value of $\min\left\{m(v), \frac{k-1}{k}\right\}$ to e'.
- **Step 2.2** Any value remaining at v is distributed equally among the child edges of v belonging to E_{r}.

Negative result for Edge-k-Coloring of Trees

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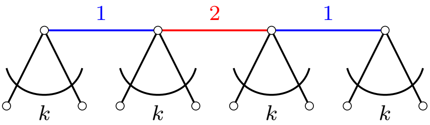

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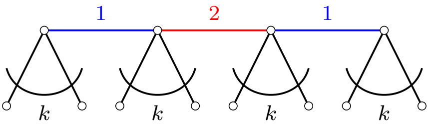

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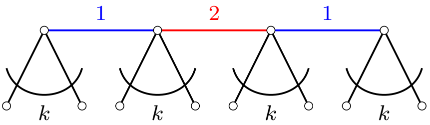

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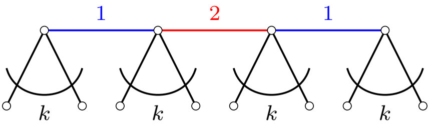

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Furthermore, one can show that the competitive ratio of Next-Fit is no better than $\frac{2\sqrt{k}-2}{2\sqrt{k}-1}$ when k is a square number.

Randomization on Trees

- First-Fit is optimal on trees among fair or deterministic algorithms with a competitive ratio of $\frac{k-1}{k}$.
- Can a randomized algorithm do better than $\frac{k-1}{k}$?

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- First-Fit is optimal on trees among fair or deterministic algorithms with a competitive ratio of $\frac{k-1}{k}$.
- Can a randomized algorithm do better than $\frac{k-1}{k}$?
- Maybe, but not better than $\frac{k}{k+1}$.

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- Trees have PA $= 1$. Planar graphs have PA at most 3.
- Graphs of bounded degree, treewidth, degeneracy or genus has bounded PA.

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Theorem

Suppose that the input graph is k-colorable and has PA at most t. If $t \leq \frac{1}{4}k$, then the competitive ratio of any fair algorithm is at least

$$\frac{2\sqrt{k/t} - 2}{2\sqrt{k/t} - 1}.$$

Parameterized Competitive Ratio

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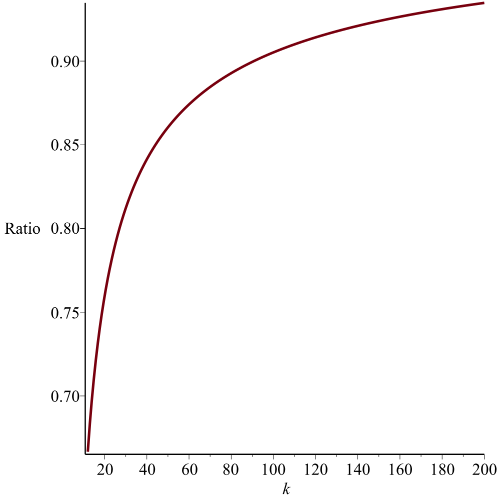
$$\frac{2\sqrt{k/t}-2}{2\sqrt{k/t}-1}.$$

The competitive ratio on k-colorable graph is also known as the competitive ratio on *accommodating sequences* [Boyar, Larsen, Nielsen '98].

Parameterized Competitive Ratio

A lower bound for any fair algorithm on planar graphs (PA ≤ 3).



Conclusion

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- First-Fit is optimal among deterministic algorithms on paths and trees.

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- First-Fit is optimal among deterministic algorithms on paths and trees.
- On tree-like graphs, any fair algorithm for online Edge-k-Coloring performs well if it has a sufficiently large number of colors.

Open Problems

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Is it possible to achieve a competitive ratio better than $2\sqrt{3}-3$ for Edge-k-Coloring?
Does First-Fit have a competitive ratio better than $2\sqrt{3}-3$ for Edge-k-Coloring?On bipartite graphs?

THANK YOU