System FOL

Language

$$\begin{array}{rcl} t & ::= & x_i \mid c_i \mid f_i(t_1, \ldots, t_{a_i} \\ \varphi, \psi & ::= & \bot \mid P_i(t_1, \ldots, t_{r_i}) \mid t_1 = t_2 \mid \varphi \to \psi \mid \forall x_i .\, \varphi \\ \Gamma & ::= & \epsilon \mid \varphi, \Gamma \end{array}$$

Derivations

$$(\mathit{assum})\ \frac{}{\Gamma \vdash^{\mathsf{FOL}} \varphi}\ \varphi \in \Gamma \qquad (\neg\neg\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{FOL}} \neg\neg\varphi}{\Gamma \vdash^{\mathsf{FOL}} \varphi}$$

$$(\to\text{-}I)\ \frac{\Gamma, \varphi \vdash^{\mathsf{FOL}} \psi}{\Gamma \vdash^{\mathsf{FOL}} (\varphi \to \psi)} \qquad (\to\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{FOL}} (\varphi \to \psi) \quad \Gamma \vdash^{\mathsf{FOL}} \varphi}{\Gamma \vdash^{\mathsf{FOL}} \psi}$$

$$(\forall\text{-}I)\ \frac{\Gamma \vdash^{\mathsf{FOL}} \varphi}{\Gamma \vdash^{\mathsf{FOL}} (\forall x_i .\, \varphi)}\ x_i \notin FV(\Gamma) \qquad (\forall\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{FOL}} (\forall x_i .\, \varphi)}{\Gamma \vdash^{\mathsf{FOL}} \varphi[x_i := t]}\ ***$$

$$(\mathit{refl})\ \frac{}{\Gamma \vdash^{\mathsf{FOL}} t = t} \qquad (\mathit{sym})\ \frac{\Gamma \vdash^{\mathsf{FOL}} t = t'}{\Gamma \vdash^{\mathsf{FOL}} t' = t} \qquad (\mathit{trans})\ \frac{\Gamma \vdash^{\mathsf{FOL}} t_1 = t_2 \quad \Gamma \vdash^{\mathsf{FOL}} t_2 = t_3}{\Gamma \vdash^{\mathsf{FOL}} t_1 = t_3}$$

$$(=\text{-}\mathit{fun})\ \frac{\Gamma \vdash^{\mathsf{FOL}} t_1 = t_1' \quad \cdots \quad \Gamma \vdash^{\mathsf{FOL}} t_{a_i} = t_{a_i}'}{\Gamma \vdash^{\mathsf{FOL}} f_i(t_1, \ldots, t_{a_i}) = f_i(t_1', \ldots, t_{a_i}')}$$

$$(=\text{-}\mathit{pred})\ \frac{\Gamma \vdash^{\mathsf{FOL}} t_1 = t_1' \quad \cdots \quad \Gamma \vdash^{\mathsf{FOL}} t_{r_i} = t_{r_i}'}{\Gamma \vdash^{\mathsf{FOL}} P_i(t_1, \ldots, t_{r_i}) \to P_i(t_1', \ldots, t_{r_i}')}$$

Semantics

A FOL-model $\mathfrak{M}$ is a tuple $\mathfrak{M} = \langle A, F, P, C \rangle$ with A a set and:

– $F = \{[\![f_1]\!]^{\mathsf{FOL}}_{\mathfrak{M}}, \ldots, [\![f_n]\!]^{\mathsf{FOL}}_{\mathfrak{M}}\}$ with $[\![f_i]\!]^{\mathsf{FOL}}_{\mathfrak{M}} : A^{a_i} \to A$;

– $P = \{[\![P_1]\!]^{\mathsf{FOL}}_{\mathfrak{M}}, \ldots, [\![P_m]\!]^{\mathsf{FOL}}_{\mathfrak{M}}\}$ with $[\![P_i]\!]^{\mathsf{FOL}}_{\mathfrak{M}} \subseteq A^{r_i}$;

– $C = \{[\![c_1]\!]^{\mathsf{FOL}}_{\mathfrak{M}}, \ldots, [\![c_k]\!]^{\mathsf{FOL}}_{\mathfrak{M}}\} \subseteq A$.

A FOL-substitution for $\mathfrak{M}$ is a function ρ that assigns a value in A to each variable x_i.

Interpretation and satisfaction

$$\begin{array}{rcl} [\![x_i]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho} & := & \rho(x_i) \\ [\![c_i]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho} & := & [\![c_i]\!]^{\mathsf{FOL}}_{\mathfrak{M}} \\ [\![f_i(t_1, \ldots, t_{a_i})]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho} & := & [\![f_i]\!]^{\mathsf{FOL}}_{\mathfrak{M}} \left([\![t_1]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho}, \ldots, [\![t_{a_i}]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho}\right) \end{array}$$

$$\begin{array}{rcl} \not\models^{\mathsf{FOL}}_{\mathfrak{M},\rho} \bot & & \\ \models^{\mathsf{FOL}}_{\mathfrak{M},\rho} P_i(t_1, \ldots, t_{r_i}) & \text{iff} & [\![P_i]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho} \left([\![t_1]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho}, \ldots, [\![t_{r_i}]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho}\right) \\ \models^{\mathsf{FOL}}_{\mathfrak{M},\rho} t_1 = t_2 & \text{iff} & [\![t_1]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho} = [\![t_2]\!]^{\mathsf{FOL}}_{\mathfrak{M},\rho} \\ \models^{\mathsf{FOL}}_{\mathfrak{M},\rho} \varphi \to \psi & \text{iff} & \not\models^{\mathsf{FOL}}_{\mathfrak{M},\rho} \varphi \text{ or } \models^{\mathsf{FOL}}_{\mathfrak{M},\rho} \psi \\ \models^{\mathsf{FOL}}_{\mathfrak{M},\rho} \forall x_i .\, \varphi & \text{iff} & \models^{\mathsf{FOL}}_{\mathfrak{M},\rho[x_i := a]} \varphi \text{ for all } a \in A \end{array}$$

Validity and consequence

(i) $\models^{\mathsf{FOL}}_{\mathfrak{M}} \varphi$ iff $\models^{\mathsf{FOL}}_{\mathfrak{M},\rho} \varphi$ for all FOL-substitutions ρ for $\mathfrak{M}$.

(ii) $\models^{\mathsf{FOL}}_{\mathfrak{M}} \Gamma$ iff $\models^{\mathsf{FOL}}_{\mathfrak{M}} \varphi$ for every $\varphi \in \Gamma$.

(iii) $\Gamma \models^{\mathsf{FOL}} \varphi$ iff $\models^{\mathsf{FOL}}_{\mathfrak{M}} \varphi$ for all FOL-models $\mathfrak{M}$ such that $\models^{\mathsf{FOL}}_{\mathfrak{M}} \Gamma$.

(iv) $\models^{\mathsf{FOL}} \varphi$ iff $\epsilon \models^{\mathsf{FOL}} \varphi$.

System D

Language

$$
\begin{array}{rcl}
t & ::= & x_i \mid c_i \mid f_i(t_1, \ldots, t_{a_i}) \mid \mathsf{if}\ \varphi\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2 \\
\varphi, \psi & ::= & \bot \mid P_i(t_1, \ldots, t_{r_i}) \mid t_1 = t_2 \mid \varphi \to \psi \mid \forall x_i.\, \varphi \\
\Gamma & ::= & \epsilon \mid \varphi, \Gamma \mid x_i, \Gamma
\end{array}
$$

Derivations

In system D the following kinds of judgements exist.

(i) A context Γ is well formed, $\Gamma \vdash^{\mathsf{D}} \mathit{wf}$.

(ii) A term t is well formed in a context Γ, $\Gamma \vdash^{\mathsf{D}} t\ \mathit{wf}$.

(iii) A formula φ is well formed in a context Γ, $\Gamma \vdash^{\mathsf{D}} \varphi\ \mathit{wf}$.

(iv) A formula φ is provable from a context Γ, $\Gamma \vdash^{\mathsf{D}} \varphi$.

Contexts:

$$
(\epsilon\text{-}\mathit{wf})\ \frac{}{\epsilon \vdash^{\mathsf{D}} \mathit{wf}} \qquad (\mathit{decl\text{-}wf})\ \frac{\Gamma \vdash^{\mathsf{D}} \mathit{wf}}{\Gamma, x_i \vdash^{\mathsf{D}} \mathit{wf}} \qquad (\mathit{assum\text{-}wf})\ \frac{\Gamma \vdash^{\mathsf{D}} \varphi\ \mathit{wf}}{\Gamma, \varphi \vdash^{\mathsf{D}} \mathit{wf}}
$$

Terms:

$$
(\mathit{var\text{-}wf})\ \frac{\Gamma \vdash^{\mathsf{D}} \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} x_i\ \mathit{wf}}\ x_i \in \Gamma \qquad (\mathit{const\text{-}wf})\ \frac{\Gamma \vdash^{\mathsf{D}} \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} c_i\ \mathit{wf}}
$$

$$
(\mathit{fun\text{-}wf})\ \frac{\Gamma \vdash^{\mathsf{D}} D_{f_i}(t_1, \ldots, t_{a_i})}{\Gamma \vdash^{\mathsf{D}} f_i(t_1, \ldots, t_{a_i})\ \mathit{wf}} \qquad (\mathit{if\text{-}wf})\ \frac{\Gamma, \vartheta \vdash^{\mathsf{D}} t_1\ \mathit{wf} \quad \Gamma, \neg\vartheta \vdash^{\mathsf{D}} t_2\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} (\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2)\ \mathit{wf}}
$$

Formulas:

$$
(\bot\text{-}\mathit{wf})\ \frac{\Gamma \vdash^{\mathsf{D}} \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} \bot\ \mathit{wf}} \qquad (\to\text{-}\mathit{wf})\ \frac{\Gamma, \varphi \vdash^{\mathsf{D}} \psi\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} (\varphi \to \psi)\ \mathit{wf}} \qquad (\forall\text{-}\mathit{wf})\ \frac{\Gamma, x_i \vdash^{\mathsf{D}} \varphi\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} (\forall x_i.\, \varphi)\ \mathit{wf}}
$$

$$
(=\text{-}\mathit{wf})\ \frac{\Gamma \vdash^{\mathsf{D}} t_1\ \mathit{wf} \quad \Gamma \vdash^{\mathsf{D}} t_2\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} t_1 = t_2\ \mathit{wf}} \qquad (\mathit{pred\text{-}wf})\ \frac{\Gamma \vdash^{\mathsf{D}} t_1\ \mathit{wf} \quad \cdots \quad \Gamma \vdash^{\mathsf{D}} t_{r_i}\ \mathit{wf} \quad \Gamma \vdash^{\mathsf{D}} \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} P_i(t_1, \ldots, t_{r_i})\ \mathit{wf}}
$$

Proofs:

$$
(\mathit{assum})\ \frac{\Gamma \vdash^{\mathsf{D}} \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} \varphi}\ \varphi \in \Gamma \qquad (\to\text{-}I)\ \frac{\Gamma, \varphi \vdash^{\mathsf{D}} \psi}{\Gamma \vdash^{\mathsf{D}} (\varphi \to \psi)} \qquad (\to\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{D}} (\varphi \to \psi) \quad \Gamma \vdash^{\mathsf{D}} \varphi}{\Gamma \vdash^{\mathsf{D}} \psi}
$$

$$
(\neg\neg\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{D}} \neg\neg\varphi}{\Gamma \vdash^{\mathsf{D}} \varphi} \qquad (\forall\text{-}I)\ \frac{\Gamma, x_i \vdash^{\mathsf{D}} \varphi}{\Gamma \vdash^{\mathsf{D}} (\forall x_i.\, \varphi)} \qquad (\forall\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{D}} (\forall x_i.\, \varphi) \quad \Gamma \vdash^{\mathsf{D}} t\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} \varphi[x_i := t]}
$$

$$(\mathit{refl})\ \frac{\Gamma \vdash^{\mathsf{D}} t\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} t = t} \quad (\mathit{sym})\ \frac{\Gamma \vdash^{\mathsf{D}} t_1 = t_2}{\Gamma \vdash^{\mathsf{D}} t_2 = t_1} \quad (\mathit{trans})\ \frac{\Gamma \vdash^{\mathsf{D}} t_1 = t_2 \quad \Gamma \vdash^{\mathsf{D}} t_2 = t_3}{\Gamma \vdash^{\mathsf{D}} t_1 = t_3}$$

$$(=\text{-}\mathit{fun})\ \frac{\Gamma \vdash^{\mathsf{D}} t_1 = t'_1 \quad \cdots \quad \Gamma \vdash^{\mathsf{D}} t_{a_i} = t'_{a_i} \quad \Gamma \vdash^{\mathsf{D}} D_{f_i}(t_1, \dots, t_{a_i}) \quad \Gamma \vdash^{\mathsf{D}} D_{f_i}(t'_1, \dots, t'_{a_i})}{\Gamma \vdash^{\mathsf{D}} f_i(t_1, \dots, t_{a_i}) = f_i(t'_1, \dots, t'_{a_i})}$$

$$(=\text{-}\mathit{pred})\ \frac{\Gamma \vdash^{\mathsf{D}} t_1 = t'_1 \quad \cdots \quad \Gamma \vdash^{\mathsf{D}} t_{r_i} = t'_{r_i} \quad \Gamma \vdash^{\mathsf{D}} \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} P_i(t_1, \dots, t_{r_i}) \to P_i(t'_1, \dots, t'_{r_i})}$$

$$(=\text{-}\mathit{if\text{-}true})\ \frac{\Gamma \vdash^{\mathsf{D}} \vartheta \quad \Gamma, \vartheta \vdash^{\mathsf{D}} t_1\ \mathit{wf} \quad \Gamma, \neg\vartheta \vdash^{\mathsf{D}} t_2\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} (\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2) = t_1}$$

$$(=\text{-}\mathit{if\text{-}false})\ \frac{\Gamma \vdash^{\mathsf{D}} \neg\vartheta \quad \Gamma, \vartheta \vdash^{\mathsf{D}} t_1\ \mathit{wf} \quad \Gamma, \neg\vartheta \vdash^{\mathsf{D}} t_2\ \mathit{wf}}{\Gamma \vdash^{\mathsf{D}} (\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2) = t_2}$$

Semantics

A D-model $\mathfrak{M}$ is a tuple $\mathfrak{M} = \langle A, F, P, C \rangle$ where:

- A, P and C are as in FOL;
- $F = \{[\![f_1]\!]^{\mathsf{D}}_{\mathfrak{M}}, \dots, [\![f_n]\!]^{\mathsf{D}}_{\mathfrak{M}}\}$ with $[\![f_i]\!]^{\mathsf{D}}_{\mathfrak{M}} : A^{a_i} \not\to A$;
- if $e_1, \dots, e_{a_i} \in A$, then $(e_1, \dots, e_{a_i}) \in [\![D_{f_i}]\!]^{\mathsf{D}}_{\mathfrak{M}}$ iff $f_i(e_1, \dots, e_{a_i})$ is defined.

A D-substitution for $\mathfrak{M}$ is a partial function that assigns values in A to *some* variables x_i.

Interpretation and satisfaction

(i) Rules for interpreting terms.

$$\begin{array}{rcl}
[\![x_i]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} & := & \rho(x_i) \\
[\![c_i]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} & := & [\![c_i]\!]^{\mathsf{D}}_{\mathfrak{M}} \\
[\![f_i(t_1, \dots, t_{a_i})]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} & := & [\![f_i]\!]^{\mathsf{D}}_{\mathfrak{M}} \left([\![t_1]\!]^{\mathsf{D}}_{\mathfrak{M},\rho}, \dots, [\![t_{a_i}]\!]^{\mathsf{D}}_{\mathfrak{M},\rho}\right) \\
[\![\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} & := & \begin{cases} [\![t_1]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} & \text{if } \models^{\mathsf{D}}_{\mathfrak{M},\rho} \vartheta \text{ and } \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_2\ \mathit{wwf} \\ [\![t_2]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} & \text{if } \models^{\mathsf{D}}_{\mathfrak{M},\rho} \neg\vartheta \text{ and } \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1\ \mathit{wwf} \end{cases}
\end{array}$$

(ii) Rules for weak well-formation of terms and formulas.

$$\begin{array}{rcl}
\models^{\mathsf{D}}_{\mathfrak{M},\rho} x_i\ \mathit{wwf} & \text{iff} & \rho(x_i) \text{ is defined} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} c_i\ \mathit{wwf} & & \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} f_i(t_1, \dots, t_{a_i})\ \mathit{wwf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1\ \mathit{wwf}, \dots, \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_{a_i}\ \mathit{wwf} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2\ \mathit{wwf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} \vartheta\ \mathit{wwf}, \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1\ \mathit{wwf} \text{ and } \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_2\ \mathit{wwf} \\
 & & \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} \bot\ \mathit{wwf} & & \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} P_i(t_1, \dots, t_{r_i})\ \mathit{wwf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1\ \mathit{wwf}, \dots, \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_{r_i}\ \mathit{wwf} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1 = t_2\ \mathit{wwf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1\ \mathit{wwf} \text{ and } \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_2\ \mathit{wwf} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} (\varphi \to \psi)\ \mathit{wwf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi\ \mathit{wwf} \text{ and } \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi\ \mathit{wwf} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} (\forall x_i.\, \varphi)\ \mathit{wwf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho[x_i:=a]} \varphi\ \mathit{wwf} \text{ for all } a \in A
\end{array}$$

(iii) For any term t, $\models^{\mathsf{D}}_{\mathfrak{M},\rho} t\ \mathit{wf}$ iff $[\![t]\!]^{\mathsf{D}}_{\mathfrak{M},\rho}$ is defined.

(iv) Rules for well formation of formulas.

$$
\begin{array}{rcl}
\models^{\mathsf{D}}_{\mathfrak{M},\rho} \bot \ \mathit{wf} & & \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} P_i(t_1,\ldots,t_{r_i}) \ \mathit{wf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1 \ \mathit{wf},\ldots,\models^{\mathsf{D}}_{\mathfrak{M},\rho} t_{r_i} \ \mathit{wf} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1 = t_2 \ \mathit{wf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1 \ \mathit{wf} \text{ and } \models^{\mathsf{D}}_{\mathfrak{M},\rho} t_2 \ \mathit{wf} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} (\varphi \to \psi) \ \mathit{wf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi \ \mathit{wf} \text{ and } \left\{ \begin{array}{l} \models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi,\ \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi \ \mathit{wf} \\ \not\models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi,\ \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi \ \mathit{wwf} \end{array} \right. \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} (\forall x_i.\, \varphi) \ \mathit{wf} & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho[x_i:=a]} \varphi \ \mathit{wf} \text{ for all } a \in A
\end{array}
$$

(v) Rules for satisfaction of formulas.

$$
\begin{array}{rcl}
\not\models^{\mathsf{D}}_{\mathfrak{M},\rho} \bot & & \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} P_i(t_1,\ldots,t_{r_i}) & \text{iff} & \left([\![t_1]\!]^{\mathsf{D}}_{\mathfrak{M},\rho},\ldots,[\![t_{r_i}]\!]^{\mathsf{D}}_{\mathfrak{M},\rho}\right) \in [\![P_i]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} t_1 = t_2 & \text{iff} & [\![t_1]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} = [\![t_2]\!]^{\mathsf{D}}_{\mathfrak{M},\rho} \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi \to \psi & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho} (\varphi \to \psi) \ \mathit{wf} \text{ and } \not\models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi \text{ or } \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi \\
\models^{\mathsf{D}}_{\mathfrak{M},\rho} \forall x_i.\, \varphi & \text{iff} & \models^{\mathsf{D}}_{\mathfrak{M},\rho[x_i:=a]} \varphi \text{ for all } a \in A
\end{array}
$$

Validity and consequence

(i) Well-formation of contexts.

(a) $\epsilon \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathit{wf}$;

(b) $\varphi, \Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathit{wf}$ iff $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi \ \mathit{wf}$ and $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathit{wf}$;

(c) $x_i, \Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathit{wf}$ iff $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho[x_i:=a]} \mathit{wf}$ for all $a \in A$.

(ii) Let $\mathcal{X}$ stand for t *wwf* or ψ *wwf*.

(a) $\epsilon \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$;

(b) $\varphi, \Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi \ \mathit{wwf}$ and $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$;

(c) $x_i, \Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho[x_i:=a]} \mathcal{X}$ for all $a \in A$.

(iii) Let $\mathcal{X}$ stand for t *wf* or ψ *wf*.

(a) $\epsilon \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$;

(b) $\varphi, \Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathcal{X}$ iff (1) $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi$ and $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$ or (2) $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \neg\varphi$ and $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathcal{X}'$ (where $\mathcal{X}'$ stands for t *wwf* or ψ *wwf*);

(c) $x_i, \Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho[x_i:=a]} \mathcal{X}$ for all $a \in A$.

(iv) Consequence.

(a) $\epsilon \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi$ iff $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi$;

(b) $\varphi, \Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi$ iff (1) $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \varphi$ and $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi$ or (2) $\models^{\mathsf{D}}_{\mathfrak{M},\rho} \neg\varphi$ and $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi \ \mathit{wwf}$;

(c) $x_i, \Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho} \psi$ iff $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\rho[x_i:=a]} \psi$ for all $a \in A$.

(v) Let $\mathcal{X}$ stand for *wf*, t *wwf*, ψ *wwf*, t *wf*, ψ *wf* or ψ. Then $\Gamma \models^{\mathsf{D}}_{\mathfrak{M}} \mathcal{X}$ iff $\Gamma \models^{\mathsf{D}}_{\mathfrak{M},\emptyset} \mathcal{X}$ and $\Gamma \models^{\mathsf{D}} \mathcal{X}$ iff $\Gamma \models^{\mathsf{D}}_{\mathfrak{M}} \mathcal{X}$ for all D-models $\mathfrak{M}$.

(vi) In particular, a formula φ is valid (denoted $\models^{\mathsf{D}} \varphi$) iff $\epsilon \models^{\mathsf{D}} \varphi$.

System T

Language

$$
\begin{array}{rcl}
t & ::= & x_i \mid c_i \mid f_i(t_1,\ldots,t_{a_i} \mid \textsf{if } \varphi \textsf{ then } t_1 \textsf{ else } t_2 \\
\varphi,\psi & ::= & \bot \mid P_i(t_1,\ldots,t_{r_i}) \mid t_1 = t_2 \mid \varphi \to \psi \mid \forall x_i.\, \varphi \\
\Gamma & ::= & \epsilon \mid \varphi, \Gamma \mid x_i, \Gamma
\end{array}
$$

Derivations

The same judgements as above.

Contexts:

$$
(\epsilon\text{-}wf)\ \frac{}{\epsilon \vdash^{\mathsf{T}} wf} \quad (decl\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} wf}{\Gamma, x_i \vdash^{\mathsf{T}} wf} \quad (assum\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} \varphi\ wf}{\Gamma, \varphi \vdash^{\mathsf{T}} wf}
$$

Terms:

$$
(var\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} x_i\ wf}\ x_i \in \Gamma \quad (const\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} c_i\ wf}
$$

$$
(fun\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} t_1\ wf \quad \cdots \quad \Gamma \vdash^{\mathsf{T}} t_{a_i}\ wf \quad \Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} f_i(t_1,\ldots,t_{a_i})\ wf} \quad (if\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} \vartheta\ wf \quad \Gamma \vdash^{\mathsf{T}} t_1\ wf \quad \Gamma \vdash^{\mathsf{T}} t_2\ wf}{\Gamma \vdash^{\mathsf{T}} (\textsf{if } \vartheta \textsf{ then } t_1 \textsf{ else } t_2)\ wf}
$$

Formulas:

$$
(\bot\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} \bot\ wf} \quad (\to\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} \varphi\ wf \quad \Gamma \vdash^{\mathsf{T}} \psi\ wf}{\Gamma \vdash^{\mathsf{T}} (\varphi \to \psi)\ wf} \quad (\forall\text{-}wf)\ \frac{\Gamma, x_i \vdash^{\mathsf{T}} \varphi\ wf}{\Gamma \vdash^{\mathsf{T}} (\forall x_i.\, \varphi)\ wf}
$$

$$
(=\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} t_1\ wf \quad \Gamma \vdash^{\mathsf{T}} t_2\ wf}{\Gamma \vdash^{\mathsf{T}} t_1 = t_2\ wf} \quad (pred\text{-}wf)\ \frac{\Gamma \vdash^{\mathsf{T}} t_1\ wf \quad \cdots \quad \Gamma \vdash^{\mathsf{T}} t_{r_i}\ wf \quad \Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} P_i(t_1,\ldots,t_{r_i})\ wf}
$$

Proofs:

$$
(assum)\ \frac{\Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} \varphi}\ \varphi \in \Gamma \quad (\to\text{-}I)\ \frac{\Gamma, \varphi \vdash^{\mathsf{T}} \psi}{\Gamma \vdash^{\mathsf{T}} (\varphi \to \psi)} \quad (\to\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{T}} (\varphi \to \psi) \quad \Gamma \vdash^{\mathsf{T}} \varphi}{\Gamma \vdash^{\mathsf{T}} \psi}
$$

$$
(\neg\neg\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{T}} \neg\neg\varphi}{\Gamma \vdash^{\mathsf{T}} \varphi} \quad (\forall\text{-}I)\ \frac{\Gamma, x_i \vdash^{\mathsf{T}} \varphi}{\Gamma \vdash^{\mathsf{T}} (\forall x_i.\, \varphi)} \quad (\forall\text{-}E)\ \frac{\Gamma \vdash^{\mathsf{T}} (\forall x_i.\, \varphi) \quad \Gamma \vdash^{\mathsf{T}} t\ wf}{\Gamma \vdash^{\mathsf{T}} \varphi[x_i := t]}
$$

$$
(refl)\ \frac{\Gamma \vdash^{\mathsf{T}} t\ wf}{\Gamma \vdash^{\mathsf{T}} t = t} \quad (sym)\ \frac{\Gamma \vdash^{\mathsf{T}} t_1 = t_2}{\Gamma \vdash^{\mathsf{T}} t_2 = t_1} \quad (trans)\ \frac{\Gamma \vdash^{\mathsf{T}} t_1 = t_2 \quad \Gamma \vdash^{\mathsf{T}} t_2 = t_3}{\Gamma \vdash^{\mathsf{T}} t_1 = t_3}
$$

$$
(=\text{-}fun)\ \frac{\Gamma \vdash^{\mathsf{T}} t_1 = t'_1 \quad \cdots \quad \Gamma \vdash^{\mathsf{T}} t_{a_i} = t'_{a_i} \quad \Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} f_i(t_1,\ldots,t_{a_i}) = f_i(t'_1,\ldots,t'_{a_i})}
$$

$$
(=\text{-}pred)\ \frac{\Gamma \vdash^{\mathsf{T}} t_1 = t'_1 \quad \cdots \quad \Gamma \vdash^{\mathsf{T}} t_{r_i} = t'_{r_i} \quad \Gamma \vdash^{\mathsf{T}} wf}{\Gamma \vdash^{\mathsf{T}} P_i(t_1,\ldots,t_{r_i}) \to P_i(t'_1,\ldots,t'_{r_i})}
$$

$$
(=\text{-}if\text{-}true)\ \frac{\Gamma \vdash^{\mathsf{T}} \vartheta \quad \Gamma \vdash^{\mathsf{T}} t_1\ wf \quad \Gamma \vdash^{\mathsf{T}} t_2\ wf}{\Gamma \vdash^{\mathsf{T}} (\textsf{if } \vartheta \textsf{ then } t_1 \textsf{ else } t_2) = t_1} \quad (=\text{-}if\text{-}false)\ \frac{\Gamma \vdash^{\mathsf{T}} \neg\vartheta \quad \Gamma \vdash^{\mathsf{T}} t_1\ wf \quad \Gamma \vdash^{\mathsf{T}} t_2\ wf}{\Gamma \vdash^{\mathsf{T}} (\textsf{if } \vartheta \textsf{ then } t_1 \textsf{ else } t_2) = t_2}
$$

Semantics

A T-model $\mathfrak{M}$ is a FOL-model. A T-substitution for $\mathfrak{M}$ is a function ρ that assigns a value in A to each variable x_i.

Interpretation and satisfaction

(i) Rules for interpreting terms.

$$
\begin{array}{rcl}
[\![x_i]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} & := & \rho(x_i) \\
[\![c_i]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} & := & [\![c_i]\!]^{\mathsf{T}}_{\mathfrak{M}} \\
[\![f_i(t_1,\ldots,t_{a_i})]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} & := & [\![f_i]\!]^{\mathsf{T}}_{\mathfrak{M}}\left([\![t_1]\!]^{\mathsf{T}}_{\mathfrak{M},\rho},\ldots,[\![t_{a_i}]\!]^{\mathsf{T}}_{\mathfrak{M},\rho}\right) \\
[\![\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} & := & \begin{cases} [\![t_1]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \vartheta \text{ and } \models^{\mathsf{T}}_{\mathfrak{M},\rho} t_2\ \mathit{wf} \\ [\![t_2]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \neg\vartheta \text{ and } \models^{\mathsf{T}}_{\mathfrak{M},\rho} t_1\ \mathit{wf} \end{cases}
\end{array}
$$

(ii) For any term t, $\models^{\mathsf{T}}_{\mathfrak{M},\rho} t\ \mathit{wf}$ iff $[\![t]\!]^{\mathsf{T}}_{\mathfrak{M},\rho}$ is defined.

(iii) Rules for well formation of formulas.

$$
\begin{array}{rcl}
\models^{\mathsf{T}}_{\mathfrak{M},\rho} \bot\ \mathit{wf} & & \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} P_i(t_1,\ldots,t_{r_i})\ \mathit{wf} & \text{iff} & \models^{\mathsf{T}}_{\mathfrak{M},\rho} t_1\ \mathit{wf},\ldots,\models^{\mathsf{T}}_{\mathfrak{M},\rho} t_{r_i}\ \mathit{wf} \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} t_1 = t_2\ \mathit{wf} & \text{iff} & \models^{\mathsf{T}}_{\mathfrak{M},\rho} t_1\ \mathit{wf} \text{ and } \models^{\mathsf{T}}_{\mathfrak{M},\rho} t_2\ \mathit{wf} \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} (\varphi \to \psi)\ \mathit{wf} & \text{iff} & \models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi\ \mathit{wf} \text{ and } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi\ \mathit{wf} \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} (\forall x_i.\, \varphi)\ \mathit{wf} & \text{iff} & \models^{\mathsf{T}}_{\mathfrak{M},\rho[x_i:=a]} \varphi\ \mathit{wf} \text{ for all } a \in A
\end{array}
$$

(iv) Rules for satisfaction of formulas.

$$
\begin{array}{rcl}
\not\models^{\mathsf{T}}_{\mathfrak{M},\rho} \bot & & \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} P_i(t_1,\ldots,t_{r_i}) & \text{iff} & \left([\![t_1]\!]^{\mathsf{T}}_{\mathfrak{M},\rho},\ldots,[\![t_{r_i}]\!]^{\mathsf{T}}_{\mathfrak{M},\rho}\right) \in [\![P_i]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} t_1 = t_2 & \text{iff} & [\![t_1]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} = [\![t_2]\!]^{\mathsf{T}}_{\mathfrak{M},\rho} \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi \to \psi & \text{iff} & \models^{\mathsf{T}}_{\mathfrak{M},\rho} (\varphi \to \psi)\ \mathit{wf} \text{ and } \not\models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi \text{ or } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi \\
\models^{\mathsf{T}}_{\mathfrak{M},\rho} \forall x_i.\, \varphi & \text{iff} & \models^{\mathsf{T}}_{\mathfrak{M},\rho[x_i:=a]} \varphi \text{ for all } a \in A
\end{array}
$$

Validity and consequence

(i) Well-formation of contexts.

(a) $\epsilon \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathit{wf}$;

(b) $\varphi, \Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathit{wf}$ iff $\models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi\ \mathit{wf}$ and $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathit{wf}$;

(c) $x_i, \Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathit{wf}$ iff $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho[x_i:=a]} \mathit{wf}$ for all $a \in A$.

(ii) Let $\mathcal{X}$ stand for $t\ \mathit{wf}$ or $\psi\ \mathit{wf}$.

(a) $\epsilon \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathcal{X}$;

(b) $\varphi, \Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi\ \mathit{wf}$ and $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathcal{X}$;

(c) $x_i, \Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \mathcal{X}$ iff $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho[x_i:=a]} \mathcal{X}$ for all $a \in A$.

(iii) Consequence.

(a) $\epsilon \models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi$ iff $\models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi$;

(b) $\varphi, \Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi$ iff (1) $\models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi$ and $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi$ or (2) $\models^{\mathsf{T}}_{\mathfrak{M},\rho} \neg\varphi$ and $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi\ \mathit{wf}$;

(c) $x_i, \Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho} \psi$ iff $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\rho[x_i:=a]} \psi$ for all $a \in A$.

(iv) Let $\mathcal{X}$ stand for wf, $t\ \mathit{wf}$, $\psi\ \mathit{wf}$ or ψ. Then $\Gamma \models^{\mathsf{T}}_{\mathfrak{M}} \mathcal{X}$ iff $\Gamma \models^{\mathsf{T}}_{\mathfrak{M},\emptyset} \mathcal{X}$ and $\Gamma \models^{\mathsf{T}} \mathcal{X}$ iff $\Gamma \models^{\mathsf{T}}_{\mathfrak{M}} \mathcal{X}$ for all T-models $\mathfrak{M}$.

(v) In particular, a formula φ is valid (denoted $\models^{\mathsf{T}} \varphi$) iff $\epsilon \models^{\mathsf{T}} \varphi$.

Auxiliary functions

From T to FOL: $\cdot^\circ$

$$
\begin{array}{rcl}
x_i & \mapsto & \{\langle \top, x_i\rangle\} \\
c_i & \mapsto & \{\langle \top, c_i\rangle\} \\
f_i(t_1,\ldots,t_{a_i}) & \mapsto & \left\{ \langle \bigwedge_{k=1}^{a_i} \psi_k, f_i(t'_1,\ldots,t'_{a_i})\rangle \mid \forall k.\, \langle \psi_k, t'_k\rangle \in {t_k}^\circ \right\} \\
(\mathsf{if}\ \varphi\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2) & \mapsto & \{\langle \varphi^\circ \wedge \psi, t'_1\rangle \mid \langle \psi, t'_1\rangle \in {t_1}^\circ\} \cup \\
 & & \quad \{\langle \neg\varphi^\circ \wedge \psi, t'_2\rangle \mid \langle \psi, t'_2\rangle \in {t_2}^\circ\}
\end{array}
$$

$$
\begin{array}{rcl}
\bot & \mapsto & \bot \\
\varphi \rightarrow \psi & \mapsto & \varphi^\circ \rightarrow \psi^\circ \\
\forall x_i.\, \varphi & \mapsto & \forall x_i.\, \varphi^\circ \\
t_1 = t_2 & \mapsto & \bigwedge_{\langle \varphi_k, t'_k\rangle \in {t_k}^\circ} (\varphi_1 \wedge \varphi_2 \rightarrow t'_1 = t'_2) \\
P_i(t_1,\ldots,t_{r_i}) & \mapsto & \bigwedge_{\langle \varphi_k, t'_k\rangle \in {t_k}^\circ} \left(\bigwedge_{k=1}^{r_i} \varphi_i \rightarrow P_i(t'_1,\ldots,t'_{r_i}) \right)
\end{array}
$$

From T to D: the $*$-functions

$$
\begin{array}{rcl}
x_i & \mapsto & x_i \\
c_i & \mapsto & c_i \\
f_i(t_1,\ldots,t_{a_i}) & \mapsto & \mathsf{if}\ D_{f_i}({t_1}^*,\ldots,{t_{a_i}}^*)\ \mathsf{then}\ f_i({t_1}^*,\ldots,{t_{a_i}}^*)\ \mathsf{else}\ c_1 \\
\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2 & \mapsto & \mathsf{if}\ \vartheta^*\ \mathsf{then}\ {t_1}^*\ \mathsf{else}\ {t_2}^*
\end{array}
$$

$$
\begin{array}{rcl}
\bot & \mapsto & \bot \\
P_i(t_1,\ldots,t_{r_i}) & \mapsto & P_i({t_1}^*,\ldots,{t_{r_i}}^*) \\
t_1 = t_2 & \mapsto & {t_1}^* = {t_2}^* \\
\varphi \rightarrow \psi & \mapsto & \varphi^* \rightarrow \psi^* \\
\forall x_i.\, \varphi & \mapsto & \forall x_i.\, \varphi^*
\end{array}
$$

This function is extended trivially to contexts: $\epsilon^* = \epsilon$, $(\Gamma, x_i)^* = \Gamma^*, x_i$ and $(\Gamma, \varphi)^* = \Gamma^*, \varphi^*$.

Let $\mathfrak{M} = \langle A, F, P, C\rangle$ be a D-model. Then $\mathfrak{M}_*$ is the T-model defined by $\mathfrak{M}_* = \langle A, F_*, P, C\rangle$, where $F_* = \{[\![f_1]\!]^{\mathsf{T}}_{\mathfrak{M}_*}, \ldots, [\![f_n]\!]^{\mathsf{D}}_{\mathfrak{M}_*}\}$ with

$$
[\![f_i]\!]^{\mathsf{T}}_{\mathfrak{M}_*}(e_1,\ldots,e_{a_i}) = \begin{cases} [\![f_i]\!]^{\mathsf{D}}_{\mathfrak{M}}(e_1,\ldots,e_{a_i}) & \text{if } [\![f_i]\!]^{\mathsf{D}}_{\mathfrak{M}}(e_1,\ldots,e_{a_i}) \text{ is defined} \\ [\![c_1]\!]^{\mathsf{D}}_{\mathfrak{M}} & \text{otherwise} \end{cases}
$$

From D to T: $\cdot_|$

Let $\mathfrak{M} = \langle A, F, P, C\rangle$ be a T-model. Then $\mathfrak{M}_|$ is the D-model defined by $\mathfrak{M}_| = \langle A, F_|, P, C\rangle$, where $F_| = \{[\![f_1]\!]^{\mathsf{D}}_{\mathfrak{M}_|}, \ldots, [\![f_n]\!]^{\mathsf{D}}_{\mathfrak{M}_|}\}$ with

$$
[\![f_i]\!]^{\mathsf{D}}_{\mathfrak{M}_|}(e_1,\ldots,e_{a_i}) = [\![f_i]\!]^{\mathsf{T}}_{\mathfrak{M}}(e_1,\ldots,e_{a_i}) \text{ if } [\![f_i]\!]^{\mathsf{T}}_{\mathfrak{M}}(e_1,\ldots,e_{a_i}) \in [\![D_{f_i}]\!]^{\mathsf{T}}_{\mathfrak{M}}
$$

Notice that, again by definition, a T-substitution for $\mathfrak{M}$ is a D-substitution for $\mathfrak{M}_|$ and vice-versa.

The domain conditions

The syntactic domain conditions

$$
\begin{aligned}
\mathcal{DC}_\Gamma(x_i) = \mathcal{DC}_\Gamma(c_i) &= \emptyset \\
\mathcal{DC}_\Gamma(f_i(t_1,\ldots,t_{a_i})) &= \mathcal{DC}_\Gamma(t_1) \cup \ldots \cup \mathcal{DC}_\Gamma(t_{a_i}) \cup \left\{\Gamma \vdash^{\mathsf{T}} D_{f_i}(t_1,\ldots,t_{a_i})\right\} \\
\mathcal{DC}_\Gamma(\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2) &= \mathcal{DC}_\Gamma(\vartheta) \cup \mathcal{DC}_{\Gamma,\vartheta}(t_1) \cup \mathcal{DC}_{\Gamma,\neg\vartheta}(t_2) \\
\\
\mathcal{DC}_\Gamma(\bot) &= \emptyset \\
\mathcal{DC}_\Gamma(P_i(t_1,\ldots,t_{r_i})) &= \mathcal{DC}_\Gamma(t_1) \cup \ldots \cup \mathcal{DC}_\Gamma(t_{r_i}) \\
\mathcal{DC}_\Gamma(t_1 = t_2) &= \mathcal{DC}_\Gamma(t_1) \cup \mathcal{DC}_\Gamma(t_2) \\
\mathcal{DC}_\Gamma(\varphi \to \psi) &= \mathcal{DC}_\Gamma(\varphi) \cup \mathcal{DC}_{\Gamma,\varphi}(\psi) \\
\mathcal{DC}_\Gamma(\forall x_i.\, \varphi) &= \mathcal{DC}_{\Gamma,x_i}(\varphi) \\
\\
\mathcal{DC}(\epsilon) &= \emptyset \\
\mathcal{DC}(\Gamma,\varphi) &= \mathcal{DC}(\Gamma) \cup \mathcal{DC}_\Gamma(\varphi) \\
\mathcal{DC}(\Gamma,x_i) &= \mathcal{DC}(\Gamma)
\end{aligned}
$$

The semantic domain conditions

$$
\begin{aligned}
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(x_i) = \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(c_i) &= \top \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(f_i(t_1,\ldots,t_{a_i})) &= \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_1) \wedge \ldots \wedge \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_{a_i}) \wedge \left([\![t_1]\!]^{\mathsf{T}}_{\mathfrak{M},\rho},\ldots,[\![t_{a_i}]\!]^{\mathsf{T}}_{\mathfrak{M},\rho}\right) \in [\![D_{f_i}]\!]^{\mathsf{T}}_{\mathfrak{M}} \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\mathsf{if}\ \vartheta\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2) &= \begin{cases} \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\vartheta) \wedge \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_1) & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \vartheta \\ \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\vartheta) \wedge \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_2) & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \neg\vartheta \end{cases} \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\bot) &= \top \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(P_i(t_1,\ldots,t_{r_i})) &= \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_1) \wedge \ldots \wedge \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_{r_i}) \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_1 = t_2) &= \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_1) \wedge \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(t_2) \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\varphi \to \psi) &= \begin{cases} \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\varphi) \wedge \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\psi) & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi \\ \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\varphi) & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \neg\varphi \end{cases} \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\forall x_i.\, \varphi) &= \bigwedge_{a\in A} \overline{\mathcal{DC}}^{\mathfrak{M},\rho[x_i:=a]}(\varphi) \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}_{\epsilon}(\mathcal{X}) &= \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\mathcal{X}) \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}_{\varphi,\Gamma}(\mathcal{X}) &= \begin{cases} \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\varphi) \wedge \overline{\mathcal{DC}}^{\mathfrak{M},\rho}_{\Gamma}(\mathcal{X}) & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \varphi \\ \overline{\mathcal{DC}}^{\mathfrak{M},\rho}(\varphi) & \text{if } \models^{\mathsf{T}}_{\mathfrak{M},\rho} \neg\varphi \end{cases} \\
\overline{\mathcal{DC}}^{\mathfrak{M},\rho}_{x_i,\Gamma}(\mathcal{X}) &= \bigwedge_{a\in A} \overline{\mathcal{DC}}^{\mathfrak{M},\rho[x_i:=a]}_{\Gamma}(\mathcal{X}) \\
\overline{\mathcal{DC}}_{\Gamma}(\mathcal{X}) &= \bigwedge_{\mathfrak{M}} \overline{\mathcal{DC}}^{\mathfrak{M},\emptyset}_{\Gamma}(\mathcal{X})
\end{aligned}
$$