The Essence of Proofs in Sequent Calculi

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Brouwer Seminar Radboud Universiteit Nijmegen 5th October 2005

- Motivation
- Examples
- Fibring
- 2 Sequent calculi given by rules
 - Definitions
 - Examples
 - Fibring
- 3 Sequent calculi given by derivations
 - Definitions
 - Fibring
 - Equivalence
- Preservation results
 - Cut elimination
 - Decidability

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- Motivation
- Examples
- Fibring
- 2 Sequent calculi given by rules
 - Definitions
 - Examples
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- 3 Sequent calculi given by derivations
 - Definitions
 - Fibring
 - Equivalence
- Preservation results
 - Cut elimination
 - Decidability
 - 5 Conclusions & future work

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- Motivation
- Examples
- Fibring
- 2 Sequent calculi given by rules
 - Definitions
 - Examples
 - Fibring
- 3 Sequent calculi given by derivations
 - Definitions
 - Fibring
 - Equivalence
- Preservation results
 - Cut elimination
 - Decidability
- 5 Conclusions & future work

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- Motivation
- Examples
- Fibring
- 2 Sequent calculi given by rules
 - Definitions
 - Examples
 - Fibring
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 - Definitions
 - Fibring
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- Preservation results
 - Cut elimination
 - Decidability
- 5 Conclusions & future work

- Motivation
- Examples
- Fibring
- 2 Sequent calculi given by rules
 - Definitions
 - Examples
 - Fibring
- 3 Sequent calculi given by derivations
 - Definitions
 - Fibring
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- Preservation results
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 - Decidability
- 5 Conclusions & future work

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

Motivation

- use of logic to describe behaviour of systems
- different systems ←→ different logics
- combination of systems

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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- different systems \longleftrightarrow different logics
- \bullet combination of systems \longleftrightarrow combination of logics

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

Two simple examples (I)

- The behaviour of system A is described by linear temporal logic with (state) propositional variables p and q.
- The behaviour of system B is described by linear temporal logic with a (state) propositional variable r.
- Under reasonable assumptions, the joint system can be described by linear temporal logic with state variables p, q and r.

Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

Two simple examples (II)

Epistemological logics (dealing with knowledge) typically include an S5 modality K.

Deontic logics (reasoning about obligation) use a D modality O.

Reasoning about Law requires the combination of these two logics, where one wants to write formulas mixing both modal operators.

 $eg KO(arphi) \wedge O(arphi) \wedge
eg arphi o (\mathsf{goto-jail})$

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Overview of fibring Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work

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 $\neg \mathcal{KO}(\varphi) \land \mathcal{O}(\varphi) \land \neg \varphi \rightarrow (\mathsf{goto-jail})$

Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

Other generic examples

parameterization of logics;

- union of logics;
- fusion of modal logics.

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

Fibring

- more generally applicable;
- fewer restrictions on language.
- Key results: preservation of properties
- syntactical: decidability, complexity
- semantical: finite model property, cardinality results; decidability

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work

Drawbacks

Motivation Examples Fibring

Propositional vs

but useful!

usually very hard

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Propositional vs

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Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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often quite simple

First-Order usually **very** hard

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4

L. Cruz-Filipe, C. Sernadas The Essence of Proofs in Sequent Calculi

Sequent calculi given by rules Sequent calculi given by derivations Preservation results Conclusions & future work Motivation Examples Fibring

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An important distinction

Homogeneous fibring deals with combining two logics presented/defined in a similar way, e.g.:

- two Hilbert calculi;
- two sequent calculi;
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Heterogeneous fibring attempts to combine two logics presented/defined by different means, e.g.:

- a Hilbert calculus and a sequent calculus;
- a sequent-calculus and a modal logic characterized by some class of Kripke structures.

Heterogeneous fibring is a much harder problem that has only recently been addressed.

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Remark: signatures

Throughout we will only consider logics with a propositional basis.

Definition

A propositional *signature* is a family $C = \{C_k\}_{k \in \mathbb{N}}$ of sets. Each $c_k \in C_k$ is called a *constructor* or *connective* of arity k.

The *language* L(C) is the free algebra over C generated by a countable set $\Xi = \{\xi_n : n \in \mathbb{N}\}$ of meta-variables.

The elements of $L(C, \Xi)$ are called *formulas*.

We say that $C \subseteq C'$ if $C_k \subseteq C'_k$ for every $k \in \mathbb{N}$.

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Definitions Examples Fibring

Definition

A sequent is a pair $\Gamma \longrightarrow \Delta$, where Γ, Δ are multisets over L(C)

A *rule* is a pair $\frac{\theta_1,\ldots,\theta_n}{\gamma}$ where $\theta_1,\ldots,\theta_n,\gamma$ are sequents.

Definition

A sequent calculus (given by rules) is a pair $\mathcal{R} = \langle C, R \rangle$, where C is a signature and R is a set of rules including structural rules and specific rules (for the connectives).

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Definitions Examples Fibring

Structural rules

These are chosen *among* the following.



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$$\begin{array}{cccc} & \underline{\xi_1, \Delta_1 \longrightarrow \Delta_2 \quad \Delta_1 \longrightarrow \Delta_2, \xi_1} \\ & \underline{\lambda_1 \longrightarrow \Delta_2} \end{array} \ \mathsf{Cut} \\ \\ & \underline{\Delta_1 \longrightarrow \Delta_2} \\ \hline & \underline{\xi_1, \Delta_1 \longrightarrow \Delta_2} \end{array} \ \mathsf{LW} & & \underline{\Delta_1 \longrightarrow \Delta_2, \xi_1} \\ \\ & \underline{\Delta_1, \xi_1, \xi_1 \longrightarrow \Delta_2} \\ \hline & \underline{\Delta_1, \xi_1, \xi_1 \longrightarrow \Delta_2} \end{array} \ \mathsf{LC} & & \underline{\Delta_1 \longrightarrow \xi_1, \xi_1, \Delta_2} \\ \hline & \underline{\Delta_1 \longrightarrow \xi_1, \Delta_2} \end{array} \ \mathsf{RC} \end{array}$$

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Definitions Examples Fibring

Rules for the connectives

These may include:

- Left rules: the antecedent of the conclusion includes a formula $c(\varphi_1, \ldots, \varphi_n)$ for some *n*-ary connective *c*.
- Right rules: the consequent of the conclusion includes a formula c(φ₁,...,φ_n) for some n-ary connective c.

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Definitions Examples Fibring

Derivations

Definition

A (rule-)derivation of a sequent s from a set of sequents Θ in sequent calculus \mathcal{R} is a finite sequence $\{\Gamma_i \longrightarrow \Delta_i\}_{i=1}^n$ of sequents such that:

- $\Gamma_1 \longrightarrow \Delta_1$ is s;
- for each i = 1, ..., n, one of the following holds:
 - $f_{1} \cap \Delta_{1} \neq \emptyset$ (justified by Ax);

 - for some substitutients $(\gamma, \{, \beta, \gamma, \dots, , \beta\}) := \gamma$ show since role •

 $[i_1 \longrightarrow \Delta_i = \sigma(\gamma)$ and $\sigma(\theta) \in [i_1 \longrightarrow \Delta_i]_{i=i_1}$

Notation: $\Delta \vdash_{\mathcal{R}} s$ or (when Δ is empty) $\vdash_{\mathcal{R}} s$.

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- for some rule r = ({θ₁,..., θ_k}, γ) and substitution σ

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 - $\Gamma_i \longrightarrow \Delta_i = \sigma(\gamma) \text{ and } \sigma(\theta_j) \in {\Gamma_k \longrightarrow \Delta_k}_{k=i+1}^n \text{ (justified by } r, i_1, \dots, i_k).$

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• $\Gamma_i \longrightarrow \Delta_i \in \Theta$ (justified by Hyp);

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Notation: $\Delta \vdash_{\mathcal{R}} s$ or (when Δ is empty) $\vdash_{\mathcal{R}} s$.

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Definitions Examples Fibring

Derivations

Definition

A (rule-)derivation of a sequent s from a set of sequents Θ in sequent calculus \mathcal{R} is a finite sequence $\{\Gamma_i \longrightarrow \Delta_i\}_{i=1}^n$ of sequents such that:

•
$$\Gamma_1 \longrightarrow \Delta_1$$
 is s;

• for each i = 1, ..., n, one of the following holds:

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Definitions Examples Fibring

Example: S4

All structural rules plus:

$$\frac{\Gamma \longrightarrow \Delta, \xi_1 \quad \xi_2, \Gamma \longrightarrow \Delta}{(\xi_1 \rightarrow \xi_2), \Gamma \longrightarrow \Delta} \ \mathsf{L} \rightarrow \quad \frac{\xi_1, \Gamma \longrightarrow \Delta, \xi_2}{\Gamma \longrightarrow \Delta, (\xi_1 \rightarrow \xi_2)} \ \mathsf{R} \rightarrow$$

$$\frac{\xi_1, \Gamma_1 \longrightarrow \Diamond(\Delta_1)}{(\Diamond \xi_1), \Box(\Gamma_1), \Gamma_2 \longrightarrow \Delta_2, \Diamond(\Delta_1)} \ \mathsf{L} \Diamond \qquad \frac{\Gamma, \xi_1, (\Box \xi_1) \longrightarrow \Delta}{\Gamma, (\Box \xi_1) \longrightarrow \Delta} \ \mathsf{L} \Box$$

 $\frac{\Box \Gamma_1 \longrightarrow \xi_1, \Delta_1}{\Gamma_2, \Box(\Gamma_1) \longrightarrow (\Box \xi_1), \Diamond(\Delta_1), \Delta_2} R \Box \qquad \frac{\Gamma \longrightarrow \Delta, \xi_1, (\Diamond \xi_1), \langle \xi_1 \rangle, \langle \xi_1$

where $\Box(\Gamma) = \{(\Box \varphi) : \varphi \in \Gamma\}$ and $\Diamond(\Gamma) = \{(\Diamond \varphi) : \varphi \in \Gamma\}$

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Definitions Examples Fibring

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Definitions Examples Fibring

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Definitions Examples Fibring

Derivation in S4

Example

The following shows that $\vdash_{S4} \longrightarrow (\Diamond(\xi_1 \rightarrow (\Box \xi_1))).$

Definitions Examples Fibring

Example: D

All structural rules plus:

$$\frac{\Gamma \longrightarrow \Delta, \xi_1 \quad \xi_2, \Gamma \longrightarrow \Delta}{(\xi_1 \rightarrow \xi_2), \Gamma \longrightarrow \Delta} \ \mathsf{L} \rightarrow \qquad \frac{\xi_1, \Gamma \longrightarrow \Delta, \xi_2}{\Gamma \longrightarrow \Delta, (\xi_1 \rightarrow \xi_2)} \ \mathsf{R} \rightarrow$$

$$\frac{\Gamma \longrightarrow \Delta, \xi_1}{\Gamma, (\neg \xi_1) \longrightarrow \Delta} \ L \neg$$

$$\frac{\Gamma, \xi_1 \longrightarrow \Delta}{\Gamma \longrightarrow (\neg \xi_1), \Delta} \ \mathsf{R} \neg$$

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$$\frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Box\xi_1)} \ \mathsf{R}\Box \qquad \frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Diamond\xi_1)} \ \mathsf{R}\Diamond$$

Definitions Examples Fibring

Example: D

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Definitions Examples Fibring

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Definitions Examples Fibring

Derivation in D

Example

The following shows that
$$\longrightarrow \xi_2 \vdash_D \longrightarrow (\Diamond(\xi_1 \rightarrow \xi_2))$$

$$\begin{array}{lll} 1. & \longrightarrow (\Diamond(\xi_1 \to \xi_2)) & \quad \text{Cut}, 2, 5 \\ 2. & (\Box\xi_2) \longrightarrow (\Diamond(\xi_1 \to \xi_2)) & \quad \mathsf{R}\Diamond, 3 \\ 3. & \xi_2 \longrightarrow (\xi_1 \to \xi_2) & \quad \mathsf{R} \to, 4 \\ 4. & \xi_2, \xi_1 \longrightarrow \xi_2 & \quad \mathsf{Ax} \\ 5. & \longrightarrow (\Diamond(\xi_1 \to \xi_2)), (\Box\xi_2) & \quad \mathsf{RW}, 6 \\ 6. & \longrightarrow (\Box\xi_2) & \quad \mathsf{R\Box}, 7 \\ 7. & \longrightarrow \xi_2 & \quad \mathsf{Hyp} \end{array}$$

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Definitions Examples Fibring

Definition

Let $\mathcal{R}' = \langle C', R' \rangle$ and $\mathcal{R}'' = \langle C'', R'' \rangle$ be sequent calculi.

The (rule-)*fibring* $\mathcal{R}' \uplus \mathcal{R}''$ of \mathcal{R}' and \mathcal{R}'' is the sequent calculus $\langle C' \cup C'', R' \cup R'' \rangle$.

Definitions Examples Fibring

Example

We can show that
$$\vdash_{S4 \uplus D} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$$

$$\begin{array}{lll} 1. & \longrightarrow \Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \text{Cut}, 2, 5 \\ 2. & (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) \to (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))) & \quad \mathsf{R} \Diamond'', 3 \\ 3. & (\Diamond'(\xi_1 \to (\Box'\xi_1))) \to (\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \mathsf{R} \to, 4 \\ 4. & \xi_2, (\Diamond'(\xi_1 \to (\Box'\xi_1))) \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \mathsf{Ax} \\ 5. & \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \mathsf{RW}, 6 \\ 6. & \longrightarrow (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \mathsf{RQ''}, 7 \\ 7. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \mathsf{R} \Diamond', 8 \\ 8. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)) & \quad \mathsf{R} \ominus', 8 \\ 9. & \xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\Box'\xi_1) & \quad \mathsf{RU''}, 10 \\ 1. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), \xi_1 \to (\Box'\xi_1)), \xi_1 & \quad \mathsf{R} \ominus, 12 \\ 2. & \xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\Box'\xi_1), \xi_1 & \quad \mathsf{Ax} \\ \end{array}$$

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Definitions Examples Fibring

The problem

There is obviously a relation between the derivation above and the ones done in S4 and D... but how can we formalize that?

"Derivation" is a derived notion, whereas rules are primitive; but useful properties (cut elimination, decidability) are properties of derivations, not of rules...

whow about taking derivations as primitive objects?

Definitions Examples Fibring

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 \rightsquigarrow how about taking *derivations* as primitive objects?

Definitions Fibring Equivalence

Definition

A sequent calculus given by derivations is a pair $\mathcal{D} = \langle C, P \rangle$ where C is a signature and $P = \{P_{\Theta} : \Theta \in \wp_{\text{fin}} \text{Seq}_C\}$ is a family of predicates $P_{\Theta} \subseteq \text{Seq}_C^* \times \text{Seq}_C$ such that the following conditions hold.

- Conclusion: if $P_{\Theta}(\omega, s)$ holds, then s is the first element in ω .
- Monotonicity: if $\Theta_1 \subseteq \Theta_2$, then $P_{\Theta_1} \subseteq P_{\Theta_2}$
- Closure under substitution: if P_Θ(ω, s) holds and σ is a substitution, then P_{σ(Θ)}(σ(ω), σ(s)) also holds.

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- Closure under substitution: if P_Θ(ω, s) holds and σ is a substitution, then P_{σ(Θ)}(σ(ω), σ(s)) also holds.

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Definitions Fibring Equivalence

Induced calculus from rules

Let $\mathcal{R} = \langle C, R \rangle$ be a sequent calculus given by rules and define $\mathcal{D}(\mathcal{R}) = \langle C, P \rangle$ where $P_{\Theta}(\omega, s)$ holds iff ω is a rule-derivation of s from Θ .

Then $\mathcal{D}(\mathcal{R})$ is a sequent calculus given by derivations.

Furthermore, $\Theta \vdash_{\mathcal{R}} s$ iff $\Theta \vdash_{\mathcal{D}(\mathcal{R})} s$.

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Definitions Fibring Equivalence

Translation

Definition

Let C and C' be signatures with $C \subseteq C'$ and $g: L(C') \rightarrow \mathbb{N}$ be an injection.

The translation $\tau_g : L(C') \rightarrow L(C)$ is a map defined inductively as follows:

- $\tau_g(\xi_i) = \xi_{2i+1}$ for $\xi_i \in \Xi$;
- $\tau_g(c(\gamma'_1, \ldots, \gamma'_k)) = c(\tau_g(\gamma'_1), \ldots, \tau_g(\gamma'_k))$ for $c \in C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C')$;
- $\tau_g(c'(\gamma'_1, \dots, \gamma'_k)) = \xi_{2g(c'(\gamma'_1, \dots, \gamma'_k))}$ for $c' \in C'_k \setminus C_k$ and $\gamma'_1, \dots, \gamma'_k \in L(C')$.

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• $au_{g}(c'(\gamma'_{1},\ldots,\gamma'_{k})) = \xi_{2g(c'(\gamma'_{1},\ldots,\gamma'_{k}))}$ for $c' \in C'_{k} \setminus C_{k}$ and $\gamma'_{1},\ldots,\gamma'_{k} \in L(C')$.

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Definitions Fibring Equivalence

Inverse translation

Definition

With C, C' and g as above, $\tau_g^{-1} : \Xi \to L(C')$ is the following substitution:

•
$$\tau_g^{-1}(\xi_{2i+1}) = \xi_i;$$

• $\tau_g^{-1}(\xi_{2i}) = g^{-1}(i).$

It is easy to check that $au^{-1}\circ au=\mathrm{id}$ and $au\circ au^{-1}=\mathrm{id}$

Definitions Fibring Equivalence

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Definitions Fibring Equivalence

Definition

Let $\mathcal{D}' = \langle C', P' \rangle$ and $\mathcal{D}'' = \langle C'', P'' \rangle$ be sequent calculi given by derivations.

The fibring $\mathcal{D}' \uplus \mathcal{D}''$ is the sequent calculus $\langle C, P \rangle$, where $C = C' \cup C''$ and each P_{Θ} is inductively defined as follows.

- if $P'_{\tau'(\Theta)}(\tau'(\omega), \tau'(s))$ holds, then $P_{\Theta}(\omega, s)$ also holds;
- if $P_{\tau''(\Theta)}''(\omega), \tau''(s))$ holds, then $P_{\Theta}(\omega, s)$ also holds;
- for finite $\Sigma = \{s_1, \ldots, s_k\} \subseteq \text{Seq}_C$, if $P_{\Theta}(\omega_i, s_i)$ holds for $i = 1, \ldots, k$ and $P_{\Sigma}(\omega_s, s)$ holds, then $P_{\Theta}(\omega, s)$ holds, where ω is the sequence of sequents $\omega_s \cdot \omega_1 \cdot \ldots \cdot \omega_k$.

' and au'' are the translations of $L(\mathcal{C})$ to $L(\mathcal{C}')$ and $L(\mathcal{C}')$

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Definitions Fibring Equivalence

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Let $\mathcal{D}' = \langle C', P' \rangle$ and $\mathcal{D}'' = \langle C'', P'' \rangle$ be sequent calculi given by derivations.

The fibring $\mathcal{D}' \uplus \mathcal{D}''$ is the sequent calculus $\langle C, P \rangle$, where $C = C' \cup C''$ and each P_{Θ} is inductively defined as follows.

- if $P'_{\tau'(\Theta)}(\tau'(\omega),\tau'(s))$ holds, then $P_{\Theta}(\omega,s)$ also holds;
- if $P_{\tau''(\Theta)}''(\omega), \tau''(s)$) holds, then $P_{\Theta}(\omega, s)$ also holds;
- for finite $\Sigma = \{s_1, \ldots, s_k\} \subseteq \text{Seq}_C$, if $P_{\Theta}(\omega_i, s_i)$ holds for $i = 1, \ldots, k$ and $P_{\Sigma}(\omega_s, s)$ holds, then $P_{\Theta}(\omega, s)$ holds, where ω is the sequence of sequents $\omega_s \cdot \omega_1 \cdot \ldots \cdot \omega_k$.

' and au'' are the translations of L(C) to L(C') and L(C'')

Definitions Fibring Equivalence

Definition

Let $\mathcal{D}' = \langle C', P' \rangle$ and $\mathcal{D}'' = \langle C'', P'' \rangle$ be sequent calculi given by derivations.

The fibring $\mathcal{D}' \uplus \mathcal{D}''$ is the sequent calculus $\langle C, P \rangle$, where $C = C' \cup C''$ and each P_{Θ} is inductively defined as follows.

- if $P'_{\tau'(\Theta)}(\tau'(\omega),\tau'(s))$ holds, then $P_{\Theta}(\omega,s)$ also holds;
- if $P_{\tau''(\Theta)}''(\omega), \tau''(s))$ holds, then $P_{\Theta}(\omega, s)$ also holds;
- for finite $\Sigma = \{s_1, \ldots, s_k\} \subseteq \text{Seq}_C$, if $P_{\Theta}(\omega_i, s_i)$ holds for $i = 1, \ldots, k$ and $P_{\Sigma}(\omega_s, s)$ holds, then $P_{\Theta}(\omega, s)$ holds, where ω is the sequence of sequents $\omega_s \cdot \omega_1 \cdot \ldots \cdot \omega_k$.

 τ' and τ'' are the translations of L(C) to L(C') and L(C'').

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 τ' and τ'' are the translations of L(C) to L(C') and L(C'').

Definitions Fibring Equivalence

Example

We show that
$$\vdash_{\mathcal{D}(S4) \uplus \mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$$

1.	$\longrightarrow (\Diamond''(\xi_1 \rightarrow \xi_2))$	Cut, 2, 5
2.	$(\Box''\xi_2) \longrightarrow (\Diamond''(\xi_1 \rightarrow \xi_2))$	R◊″, 3
3.	$\xi_2 \longrightarrow (\xi_1 \rightarrow \xi_2)$	$R \to, 4$
4.	$\xi_2, \xi_1 \longrightarrow \xi_2$	Ax
5.	$\longrightarrow (\Diamond''(\xi_1 \rightarrow \xi_2)), (\Box''\xi_2)$	RW, 6
6.	$\longrightarrow (\Box''\xi_2)$	$R\Box'',7$
7	. <i>F</i>	11
1.	$\longrightarrow \xi_2$	пур
<i>1</i> .	$\xrightarrow{\longrightarrow} \xi_2$ $\longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))$	Hyp R◊′,2
1. 2.		$\frac{R}{R}$
1. 2. 3.		$\frac{R p}{R \diamond', 2}$ $R \to, 3$ $R \Box', 4$
1. 2. 3. 4.		$\begin{array}{c} Hyp \\ R\Diamond', 2 \\ R \to, 3 \\ R\Box', 4 \\ R\Diamond', 5 \end{array}$
1. 2. 3. 4. 5.		$ \begin{array}{c} Hyp \\ R \Diamond', 2 \\ R \to, 3 \\ R \Box', 4 \\ R \Diamond', 5 \\ R \to, 6 \end{array} $

L. Cruz-Filipe, C. Sernadas The Essence of Proofs in Sequent Calculi

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Fibring Equivalence

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4.	$\xi_2, \xi_1 \longrightarrow \xi_2$	Ax
5.	$\longrightarrow (\Diamond''(\xi_1 \rightarrow \xi_2)), (\Box''\xi_2)$	RW, 6
6.	$\longrightarrow (\Box''\xi_2)$	$R\Box'', 7$
7.	$\longrightarrow \xi_2$	Нур
1.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	R◊′,2
2.	$\longrightarrow (\Diamond'(\xi_1 ightarrow (\Box'\xi_1))), (\xi_1 ightarrow (\Box'\xi_1))$	R ightarrow, 3
3.	$\epsilon = (\wedge' (\epsilon = (\Box' \epsilon))) (\Box' \epsilon)$	
	$\zeta_1 \longrightarrow (\lor (\zeta_1 \rightarrow (\sqcup \zeta_1))), (\sqcup \zeta_1)$	κ⊔,4
4.	$ \begin{array}{c} \zeta_1 \longrightarrow (\lor (\zeta_1 \rightarrow (\Box \ \zeta_1))), (\Box \ \zeta_1) \\ \longrightarrow (\diamondsuit'(\xi_1 \rightarrow (\Box'\xi_1))), \xi_1 \end{array} $	R⊔ ,4 R◊′,5
4. 5.	$ \begin{array}{c} \varsigma_1 \longrightarrow (\Diamond (\varsigma_1 \rightarrow (\Box \ \varsigma_1))), (\Box \ \varsigma_1) \\ \longrightarrow (\Diamond' (\xi_1 \rightarrow (\Box' \xi_1))), \xi_1 \\ \longrightarrow (\Diamond' (\xi_1 \rightarrow (\Box' \xi_1))), (\xi_1 \rightarrow (\Box' \xi_1)), \xi_1 \end{array} $	R⊔ ,4 R◊′,5 R →,6

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Example

We show that
$$\vdash_{\mathcal{D}(S4) \uplus \mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$$

$$\begin{array}{cccc} 1. & \longrightarrow (\Diamond''(\xi_1 \to (\bigcirc'(\xi_1 \to (\Box'\xi_1))))) & \quad & \text{Cut}, 2, 5 \\ 2. & (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) \longrightarrow (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))) & \quad & \text{R} \Diamond'', 3 \\ 3. & (\Diamond'(\xi_1 \to (\Box'\xi_1))) \longrightarrow (\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad & \text{R} \to, 4 \\ 4. & (\Diamond'(\xi_1 \to (\Box'\xi_1))), \xi_1 \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad & \text{Ax} \\ 5. & \longrightarrow (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad & \text{RW}, 6 \\ 6. & \longrightarrow (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad & \text{RU}', 7 \\ 7. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad & \text{RU}', 7 \\ 1. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)) & \quad & \text{RO}', 2 \\ 2. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)) & \quad & \text{RU}', 4 \\ 4. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)), \xi_1 & \quad & \text{RO}', 5 \\ 5. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)), \xi_1 & \quad & \text{R} \to, 6 \\ 6. & \xi_1 \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\Box'\xi_1), \xi_1 & \quad & \text{Ax} \\ \end{array}$$

Definitions Fibring Equivalence

Example

We show that
$$\vdash_{\mathcal{D}(54) \uplus \mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$$

$$\begin{array}{lll} 1. & \longrightarrow (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))) & \quad \text{Cut}, 2, 5 \\ 2. & (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) \to (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))) & \quad \text{R} \Diamond'', 3 \\ 3. & (\Diamond'(\xi_1 \to (\Box'\xi_1))) \to (\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \text{R} \to, 4 \\ 4. & (\Diamond'(\xi_1 \to (\Box'\xi_1))), \xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \text{Ax} \\ 5. & \longrightarrow (\Diamond''(\xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \text{RW}, 6 \\ 6. & \longrightarrow (\Box''(\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \text{RU''}, 7 \\ 7. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1)))) & \quad \text{RD''}, 7 \\ 7. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))) & \quad \text{RO''}, 2 \\ 2. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)) & \quad \text{RO''}, 4 \\ 3. & \xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)), \xi_1 & \quad \text{RO''}, 5 \\ 5. & \longrightarrow (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\xi_1 \to (\Box'\xi_1)), \xi_1 & \quad \text{R} \to, 6 \\ 6. & \xi_1 \to (\Diamond'(\xi_1 \to (\Box'\xi_1))), (\Box'\xi_1), \xi_1 & \quad \text{Ax} \end{array}$$

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Theorem

Let $\mathcal{R}' = \langle C', R' \rangle$ and $\mathcal{R}'' = \langle C'', R'' \rangle$ be sequent calculi given by rules such that Cut, LW and RW are in $R' \cup R''$, and define:

- \$\mathcal{D}' = \mathcal{D}(\mathcal{R}')\$ and \$\mathcal{D}'' = \mathcal{D}(\mathcal{R}'')\$ are the sequent calculi given by derivations induced by \$\mathcal{R}'\$ and \$\mathcal{R}''\$;
- $\mathcal{R} = \mathcal{R}' \uplus \mathcal{R}''$ is the fibring of \mathcal{R}' and \mathcal{R}'' ;
- $\mathcal{D} = \mathcal{D}' \uplus \mathcal{D}''$ is the fibring of \mathcal{D}' and \mathcal{D}'' ;
- $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ is the common signature of \mathcal{R} and \mathcal{D} .

Then \mathcal{D} and \mathcal{R} are equivalent systems in the sense that $\Delta \vdash_{\mathcal{R}} s$ iff $\Delta \vdash_{\mathcal{D}} s$, for any $\Delta \subseteq \operatorname{Seq}_{\mathcal{C}}$ and $s \in \operatorname{Seq}_{\mathcal{C}}$.

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Cut elimination Decidability

Definition

A sequent calculus given by rules $\mathcal{R} = \langle C, R \rangle$ has cut elimination iff, for any $\Delta \subseteq \text{Seq}_C$ and $s \in \text{Seq}_C$, whenever $\Delta \vdash_{\mathcal{R}} s$ there is a derivation ω for $\Delta \vdash_{\mathcal{R}} s$ that does not use the cut rule.

Theorem

Let \mathcal{R}' and \mathcal{R}'' be sequent calculi given by rules with cut elimination. Then their fibring \mathcal{R} also has cut elimination.

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Cut elimination Decidability

Definition

A sequent calculus given by derivations $\mathcal{D} = \langle C, P \rangle$ is *decidable* iff, for every recursive set $\Delta \subseteq \text{Seq}_C$, the relation P_Δ is recursive.

A sequent calculus given by rules \mathcal{R} is decidable iff $\mathcal{D}(\mathcal{R})$ is decidable.

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A sequent calculus given by rules ${\mathcal R}$ is decidable iff ${\mathcal D}({\mathcal R})$ is decidable.

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Cut elimination Decidability

Theorem (Characterization via rules)

A \mathcal{R} be a sequent calculus given by rules is decidable iff for every rule r the relation S_r is recursive, where S_r is the relation such that $S_r(s_1, \ldots, s_n, s)$ holds iff $\langle \{s_1, \ldots, s_n\}, s \rangle$ is an instance of r.

Corollary

Let \mathcal{R}' and \mathcal{R}'' be decidable sequent calculi given by rules.

Then their fibring $\mathcal{R} = \mathcal{R}' \uplus \mathcal{R}''$ is decidable.

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Let \mathcal{D}' and \mathcal{D}'' be decidable sequent calculi given by derivations.

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Cut elimination Decidability

Algorithm

- For each partition of ω do
 - If the partition is singular, check whether
 P'_{τ'(Δ)}(τ'(ω), τ'(s)) holds or P''_{τ''(Δ)}(τ''(ω), τ''(s)) holds.
 If either is the case, output 1; otherwise move to the next partition.
 - Otherwise, let ω^* be the first sequence in the partition and $\omega_1, \ldots, \omega_n$ the remaining ones. Let s_i denote $(\omega_i)_i$
 - For each i = 1,...., n check whether P_Λ(ω₀, s_i) holds, this is not the case, go on to the next partition.
 - If the test above succeeded for all *i*, check whether
 - $P_{[\alpha_1,\dots,\alpha_l]}(\omega, s)$ holds. If this is the case, output 1.

• When no partitions of ω are left, output 0.

Cut elimination Decidability

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 - If the partition is singular, check whether
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 - For each i = 1,..., n check whether P_Δ(ω_i, s_i) holds. If this is not the case, go on to the next partition.
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 - If the test above succeeded for all *i*, check whether P_{{s1},...,s_n}(ω, s) holds. If this is the case, output 1.
- When no partitions of ω are left, output 0.

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Cut elimination Decidability

Algorithm

- For each partition of ω do
 - If the partition is singular, check whether $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds or $P''_{\tau''(\Delta)}(\tau''(\omega), \tau''(s))$ holds. If either is the case, output 1; otherwise move to the next partition.
 - Otherwise, let ω^{*} be the first sequence in the partition and ω₁,..., ω_n the remaining ones. Let s_i denote (ω_i)₁.
 - For each i = 1,..., n check whether P_Δ(ω_i, s_i) holds. If this is not the case, go on to the next partition.
 - If the test above succeeded for all *i*, check whether
 P_{{s1} = s₁}(ω, s) holds. If this is the case, output 1.
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 - So For each i = 1,..., n check whether P_Δ(ω_i, s_i) holds. If this is not the case, go on to the next partition.
 - If the test above succeeded for all *i*, check whether P_{s1,...,sn}(ω, s) holds. If this is the case, output 1.

• When no partitions of ω are left, output 0.

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• nice :-) definition of sequent calculus via derivations

- new definition of fibring for sequent calculi
- preservation of cut-elimination
- preservation of decidability

Future work

- generalization of the notion of sequent:
- generalization beyond propositional signature.

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