## Reasoning about Probabilistic Sequential Programs

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## Motivation

- reasoning about non-deterministic programs
- new approach: truth values for formulas



#### • reasoning about non-deterministic programs

new approach: truth values for formulas

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- reasoning about non-deterministic programs
- new approach: truth values for formulas

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- Language
- Semantics
- Calculus
- Properties
- 2 The Programming Language
  - Syntax
  - Semantics
- 3 The Hoare Calculus
  - The calculus
  - Soundness
  - Completeness



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## 4 Conclusions

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Language Semantics Calculus Properties

## Why EPPL

### • two-layered design (exogenous approach)

- classical propositional logic at the lower level
- probabilistic logic built at the higher level

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Language Semantics Calculus Properties



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Language Semantics Calculus Properties

## Real-closed fields

#### Definition

A real closed field is an ordered field  ${\cal K}$  where:

- every non-negative element of the K has a square root in K;
- every polynomial of odd degree with coefficients in K has at least one solution in K.

#### Example

- the set of real numbers with the usual multiplication, addition and order relation;
- the set of computable real numbers with the same operations.

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Language Semantics Calculus Properties

# Setting

### • finite range D of real numbers

- finite set  $\mathbf{m} = \{0, \dots, m-1\}$  of indices
- registers  $\mathbf{xM} = {\mathbf{xm}_k \mid k \in \mathbf{m}}$  containing real values
- registers  $\mathbf{bM} = {\mathbf{bm}_k \mid k \in \mathbf{m}}$  containing booleans
- variables  $B = \{B_k : k \in \mathbb{N}\}$  ranging over truth values
- variables  $X = \{X_k : k \in \mathbb{N}\}$  ranging over D
- ullet real-closed field  ${\cal K}$  with set of algebraic numbers  ${\cal A}$
- logical variables  $Y = \{y_k : k \in \mathbb{N}\}$  ranging over  $\mathcal{K}$

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Language Semantics Calculus Properties

## Language

Real terms (with  $c \in D$ )

 $t ::= c \mid \mathbf{xm} \mid X \mid (t+t) \mid (t t)$ 

Classical state formulas

 $\gamma ::= \mathsf{bm} \mid B \mid (t \leq t) \mid \mathsf{ff} \mid (\gamma \Rightarrow \gamma)$ 

Probability terms (with  $r \in \mathcal{A}$ )  $p ::= r \mid y \mid \widetilde{r} \mid (\int \gamma) \mid (p + p) \mid (p p)$ 

Probabilistic state formulas

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 The State Logic: EPPL
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Language Semantics Calculus Properties

# Useful notions

#### Definition

An *analytical term* is a term without occurrences of probability terms.

$$a ::= r \mid y \mid \widetilde{r} \mid (a+a) \mid (aa)$$

#### Definition

An *analytical formula* is a formula without occurrences of probability terms.

$$\kappa ::= (a \le a) \mid \text{fff} \mid (\kappa \supset \kappa)$$

 $(\Box\gamma)$  stands for the formula  $((\int\gamma) = (\int t))$  $(\Diamond\gamma)$  stands for the formula  $(\ominus(\Box(\neg\gamma)))$ 

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Language Semantics Calculus Properties

# Valuations

### Definition

A valuation is a map that provides values to the memory variables and corresponding logical variables. The set of all valuations is denoted by  $\mathcal{V}$ .

The denotation  $\llbracket t \rrbracket_{v}$  of a real term t given a valuation v is defined inductively as expected. Satisfaction  $v \Vdash_{c} \gamma$  of a classical state formula  $\gamma$  by a valuation v is also defined inductively as usual.

#### Definition

The *extent* of a classical state formula  $\gamma$  in a set V of valuations is

## $|\gamma|_V = \{ v \in V \mid v \Vdash_{\mathsf{c}} \gamma \}.$

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Language Semantics Calculus Properties

## Measure functions

#### Definition

A finitely additive, discrete and bounded  $\mathcal{K}$ -measure  $\mu$  on a set X is a map from X to  $\mathcal{K}^+$  such that:

•  $\mu(\emptyset) = 0;$ • if  $U_1 \cap U_2 = \emptyset$ , then  $\mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2).$ 

A  ${\cal K}$ -measure  $\mu$  over X is a probability measure if  $\mu(X)=1.$ 

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Language Semantics Calculus Properties

## Interpretation

#### Definition

A generalized probabilistic state consists of a real closed field  $\mathcal{K}$  and a finitely additive, discrete and finite  $\mathcal{K}$ -measure over  $\wp \mathcal{V}$ .

Given a classical formula  $\gamma$  we define

 $\mu_{\gamma} = \lambda V.\mu(|\gamma|_V).$ 

#### Definition

Given a real closed field  $\mathcal{K}$ , a  $\mathcal{K}$ -assignment is a map  $\rho: Y \to \mathcal{K}$ .

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Language Semantics Calculus Properties

## Interpretation

#### Denotation of probability terms

$$\begin{bmatrix} r \end{bmatrix}_{K,\mu}^{\rho} = r \\ \begin{bmatrix} y \end{bmatrix}_{K,\mu}^{\rho} = \rho(y) \\ \begin{bmatrix} (\int \gamma) \end{bmatrix}_{K,\mu}^{\rho} = \mu(|\gamma|_{\mathcal{V}}) \\ \begin{bmatrix} p_{1} + p_{2} \end{bmatrix}_{K,\mu}^{\rho} = \begin{bmatrix} p_{1} \end{bmatrix}_{K,\mu}^{\rho} + \begin{bmatrix} p_{2} \end{bmatrix}_{K,\mu}^{\rho} \\ \begin{bmatrix} p_{1} p_{2} \end{bmatrix}_{K,\mu}^{\rho} = \begin{bmatrix} p_{1} \end{bmatrix}_{K,\mu}^{\rho} \times \begin{bmatrix} p_{2} \end{bmatrix}_{K,\mu}^{\rho}$$

Satisfaction of probabilistic formulas  $(K,\mu)
ho \Vdash (
ho_1 \le 
ho_2)$  iff  $\llbracket p_1 \rrbracket_{K,\mu}^{
ho} \le \llbracket p_2 \rrbracket_{K,\mu}^{
ho}$   $(K,\mu)
ho \nVdash$ fff  $(K,\mu)
ho \Vdash (\eta_1 \supset \eta_2)$  iff  $(K,\mu)
ho \Vdash \eta_2$  or  $(K,\mu)
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Denotation of probability terms

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Satisfaction of probabilistic formulas  $(K,\mu)\rho \Vdash (p_1 \le p_2) \quad \text{iff} \quad \llbracket p_1 \rrbracket_{K,\mu}^{\rho} \le \llbracket p_2 \rrbracket_{K,\mu}^{\rho}$   $(K,\mu)\rho \Vdash \# \text{fff}$   $(K,\mu)\rho \Vdash (\eta_1 \supset \eta_2) \quad \text{iff} \quad (K,\mu)\rho \Vdash \eta_2 \text{ or } (K,\mu)\rho$ 

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Language Semantics Calculus Properties

## Interpretation

Denotation of probability terms

$$\begin{bmatrix} r \end{bmatrix}_{K,\mu}^{\rho} = r \\ \begin{bmatrix} y \end{bmatrix}_{K,\mu}^{\rho} = \rho(y) \\ \begin{bmatrix} (\int \gamma) \end{bmatrix}_{K,\mu}^{\rho} = \mu(|\gamma|_{\mathcal{V}}) \\ \begin{bmatrix} p_{1} + p_{2} \end{bmatrix}_{K,\mu}^{\rho} = \begin{bmatrix} p_{1} \end{bmatrix}_{K,\mu}^{\rho} + \begin{bmatrix} p_{2} \end{bmatrix}_{K,\mu}^{\rho} \\ \begin{bmatrix} p_{1} p_{2} \end{bmatrix}_{K,\mu}^{\rho} = \begin{bmatrix} p_{1} \end{bmatrix}_{K,\mu}^{\rho} \times \begin{bmatrix} p_{2} \end{bmatrix}_{K,\mu}^{\rho}$$

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

## Interpretation

Denotation of probability terms

$$[[r]]_{K,\mu}^{\rho} = r [[y]]_{K,\mu}^{\rho} = \rho(y) [[(\int \gamma)]]_{K,\mu}^{\rho} = \mu(|\gamma|_{\mathcal{V}}) [[p_1 + p_2]]_{K,\mu}^{\rho} = [[p_1]]_{K,\mu}^{\rho} + [[p_2]]_{K,\mu}^{\rho} [[p_1 p_2]]_{K,\mu}^{\rho} = [[p_1]]_{K,\mu}^{\rho} \times [[p_2]]_{K,\mu}^{\rho}$$

Satisfaction of probabilistic formulas

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

## Interpretation

Denotation of probability terms

Satisfaction of probabilistic formulas

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Language Semantics Calculus Properties

## Auxiliary notions

#### Definition

A classical state formula  $\gamma$  is said to be *valid* if it holds for all valuations  $v \in \mathcal{V}$ .

#### Example

$$((\mathtt{x1} \le \mathtt{x2}) \land (\mathtt{x1} > 0)) \Rightarrow (\mathtt{x1}^2 \le \mathtt{x2}^2)$$

Since D is finite, the set of valid classical state formulas is recursive.

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

## Auxiliary notions

#### Definition

A probabilistic formula  $\eta$  is said to be a *probabilistic tautology* if there exists a propositional tautology  $\beta$  such that  $\eta$  is obtained from  $\beta$  by replacing all occurrences of  $\bot$  by fff,  $\rightarrow$  by  $\supset$  and each propositional symbol (uniformly) by a probabilistic state formula.

Example  $((\int (x_1 \le x_2)) < 1) \supset (((\int (x_1 \le x_2)) < 1) \cap \texttt{tt})$ 

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

## Auxiliary notions

#### Definition

An analytical formula  $\kappa$  is a *valid analytical formula* if  $\kappa$  is satisfied by  $\rho$  for any real closed field  $\mathcal{K}$  and any  $\mathcal{K}$ -assignment  $\rho$ .

#### Example

 $((y_1 \le y_2) \land (y_1 > 0)) \supset (y_1^2 \le y_2^2)$ 

The set of valid analytical formulas is decidable.

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

## Calculus

#### Axioms

- $[ extsf{CTaut}] \hspace{0.2cm} \vdash \hspace{0.2cm} (\Box\gamma)$  for each valid state formula  $\gamma$
- $[{f PTaut}] \hspace{.1in} \vdash \hspace{.1in} \eta$  for each probabilistic tautology  $\eta$ 
  - $[\mathbf{RCF}] \vdash \kappa_{\vec{p}}^{\vec{y}}$  for any valid analytical formula  $\kappa$

# $$\begin{split} [\mathsf{Meas}\emptyset] &\vdash ((\int \mathrm{ff}) = 0) \\ [\mathsf{FAdd}] &\vdash (((\int (\gamma_1 \land \gamma_2)) = 0) \supset ((\int (\gamma_1 \lor \gamma_2)) = (\int \gamma_1) + (\int \gamma_2))) \\ [\mathsf{Mon}] &\vdash ((\Box(\gamma_1 \Rightarrow \gamma_2)) \supset ((\int \gamma_1) \le (\int \gamma_2))) \end{split}$$

Inference rule

### $[\mathsf{PMP}] \quad \eta_1, (\eta_1 \supset \eta_2) \vdash \eta_2$

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Language Semantics Calculus Properties

## Calculus

#### Axioms

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Language Semantics Calculus Properties

## Calculus

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#### Language Semantics Calculus Properties

## Calculus

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Language Semantics Calculus Properties

## Soundness

#### Theorem

The axiom system of EPPL is sound: if  $\vdash \eta$ , then  $\models \eta$ .

#### Proof.

Straightforward from the definition of the semantics.

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

## Completeness and Decidability

#### Theorem

The proof system of EPPL is weakly complete: if  $\vDash \eta$ , then  $\vdash \eta$ . Moreover, the set of theorems of EPPL is recursive.

#### Proof.

The central result is to show that if  $\eta$  is consistent then there is a model  $(\mathcal{K}, \mu)\rho$  such that  $(\mathcal{K}, \mu)\rho \Vdash \eta$ . The decidability follows by showing that the consistency of a formula is decidable.

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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

## Construction of the model

- compute the (finite) set of valuations over the memory cells and the logical variables in the sets B and X occurring in η and let this set of valuations be V;
- 2 let  $\kappa_1$  be the analytical formula obtained from  $\eta$  by effectively replacing measure terms  $(\int \gamma)$  by sums  $\sum_{\nu \Vdash_c \gamma, \nu \in V} y_{\nu}$  where  $y_{\nu}$  represents the probability of the valuation  $\nu$ ;
- 3 let  $\kappa$  be the analytical formula  $\kappa_1 \cap \bigcap_{y_v | v \in V} (0 \le y_v);$
- $\eta$  is consistent iff  $\kappa$  is;
- finally, consistency of κ is decided by the axiom RCF and the model is constructed for a consistent κ by solving for y<sub>ν</sub> in real closed fields.

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Language Semantics Calculus Properties

## Construction of the model

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- **③** let  $\kappa$  be the analytical formula  $\kappa_1 \cap \bigcap_{v_v | v \in V} (0 \le y_v)$ ;
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Language Semantics Calculus Properties

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Language Semantics Calculus Properties

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Syntax Semantics

## Syntax

### $s ::= \text{skip} \mid \mathbf{xm} \leftarrow t \mid \mathbf{bm} \leftarrow \gamma \mid \text{toss}(\mathbf{bm}, r) \mid s; s \mid \text{if } \gamma \text{ then } s \text{ else } s$

#### Definition

An *expression* is either a term t or a classical state formula  $\gamma$ .

Expressions may contain variables in the set X (input to the program).

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Syntax Semantics

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Syntax Semantics

## Notation

## $\llbracket \gamma \rrbracket_{v} = \texttt{tt} \text{ if } v \Vdash_{\mathsf{c}} \gamma \text{ and } \llbracket \gamma \rrbracket_{v} = \texttt{ff} \text{ otherwise}$

if m is a memory cell and e is an expression of the same type, then  $\delta_e^m(v)$  assigns the value  $[\![e]\!]_v$  to the cell m and coincides with v elsewhere

 $(\mathcal{K}, \mu_1) + (\mathcal{K}, \mu_2) = (\mathcal{K}, \mu_1 + \mu_2)$  $r(\mathcal{K}, \mu) = (\mathcal{K}, r\mu)$ 

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Syntax Semantics

## Denotation of programs

The denotation of a program s is a function on generalized probabilistic states.

 $\begin{bmatrix} \operatorname{skip} \end{bmatrix} = \lambda(\mathcal{K},\mu).(\mathcal{K},\mu) \\ \begin{bmatrix} \operatorname{xm} \leftarrow t \end{bmatrix} = \lambda(\mathcal{K},\mu).(\mathcal{K},\mu \circ (\delta_t^{\operatorname{xm}})^{-1}) \\ \begin{bmatrix} \operatorname{bm} \leftarrow \gamma \end{bmatrix} = \lambda(\mathcal{K},\mu).(\mathcal{K},\mu \circ (\delta_{\gamma}^{\operatorname{bm}})^{-1}) \\ \begin{bmatrix} \operatorname{toss}(\operatorname{bm},r) \end{bmatrix} = \lambda(\mathcal{K},\mu).(\widetilde{r}(\llbracket\operatorname{bm} \leftarrow t \rrbracket (\mathcal{K},\mu)) + (1-\widetilde{r})(\llbracket\operatorname{bm} \leftarrow \operatorname{ff} \rrbracket (\mathcal{K},\mu))) \\ \begin{bmatrix} \operatorname{s}_1; \operatorname{s}_2 \end{bmatrix} = \lambda(\mathcal{K},\mu).[\operatorname{s}_2](\llbracket\operatorname{s}_1 \rrbracket (\mathcal{K},\mu)) \\ \stackrel{\mathsf{f}}{\gamma} \text{ then } \operatorname{s}_1 \text{ else } \operatorname{s}_2 \end{bmatrix} = \lambda(\mathcal{K},\mu).(\llbracket\operatorname{s}_1 \rrbracket (\mathcal{K},\mu_{\gamma}) + \llbracket\operatorname{s}_2 \rrbracket (\mathcal{K},\mu_{(-\gamma)})) \end{aligned}$ 

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Syntax Semantics

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The denotation of a program s is a function on generalized probabilistic states.

$$\begin{split} \begin{bmatrix} \mathsf{skip} \end{bmatrix} &= \lambda(\mathcal{K},\mu).(\mathcal{K},\mu) \\ \llbracket \mathsf{xm} \leftarrow t \rrbracket &= \lambda(\mathcal{K},\mu).(\mathcal{K},\mu \circ (\delta_t^{\mathsf{xm}})^{-1}) \\ \llbracket \mathsf{bm} \leftarrow \gamma \rrbracket &= \lambda(\mathcal{K},\mu).(\mathcal{K},\mu \circ (\delta_{\gamma}^{\mathsf{bm}})^{-1}) \\ \llbracket \mathsf{toss}(\mathsf{bm},r) \rrbracket &= \lambda(\mathcal{K},\mu).(\widetilde{r}(\llbracket \mathsf{bm} \leftarrow \mathsf{tt} \rrbracket (\mathcal{K},\mu)) + (1-\widetilde{r})(\llbracket \mathsf{bm} \leftarrow \mathsf{ft} \rrbracket (\mathcal{K},\mu))) \\ \llbracket \mathsf{s}_1,\mathsf{s}_2 \rrbracket &= \lambda(\mathcal{K},\mu).[\llbracket \mathsf{s}_2 \rrbracket (\llbracket \mathsf{s}_1 \rrbracket (\mathcal{K},\mu))) \\ \llbracket \mathsf{s}_1,\mathsf{s}_2 \rrbracket &= \lambda(\mathcal{K},\mu).[\llbracket \mathsf{s}_2 \rrbracket (\llbracket \mathsf{s}_1 \rrbracket (\mathcal{K},\mu)) + \llbracket \mathsf{s}_2 \rrbracket (\mathcal{K},\mu_{(-\gamma)})) \end{split}$$

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Syntax Semantics

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The calculus Soundness Completeness

### Hoare assertions

## $\Psi ::= \eta \mid \{\eta\} \, \mathbf{s} \, \{\eta\}$

## $(\mathcal{K},\mu)\rho \Vdash_{h} \eta \quad \text{if} \quad (\mathcal{K},\mu)\rho \Vdash \eta$ $(\mathcal{K},\mu)\rho \Vdash_{h} \{\eta_{1}\} s \{\eta_{2}\} \quad \text{if} \quad (\mathcal{K},\mu)\rho \Vdash \eta_{2} \text{ whenever } \llbracket s \rrbracket (\mathcal{K},\mu)\rho \Vdash \eta_{1}$

#### Definition

A Hoare assertion  $\Psi$  is *semantically valid* ( $\vDash_h \Psi$ ) if  $(\mathcal{K}, \mu)\rho \Vdash_h \Psi$ for every generalized probabilistic state  $(\mathcal{K}, \mu)$  and any  $\mathcal{K}$ -assignment  $\rho$ .

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

## Tossed terms

## Let **bm** be a memory cell, $r \in A$ be a constant and p be a probabilistic term.

The term toss(**bm**, *r*; *p*) is the term obtained from *p* by replacing every occurrence of each measure term  $(\int \gamma)$  by  $\tilde{r}(\int \gamma_{tt}^{bm}) + (1 - \tilde{r})(\int \gamma_{ft}^{bm})$ .

$$\begin{aligned} & \operatorname{toss}(\mathbf{bm}, r; r') &= r' \\ & \operatorname{toss}(\mathbf{bm}, r; y) &= y \\ & \operatorname{toss}(\mathbf{bm}, r; (\int \gamma)) &= (\widetilde{r}(\int \gamma_{\mathrm{tt}}^{\mathrm{bm}}) + (1 - \widetilde{r})(\int \gamma_{\mathrm{ff}}^{\mathrm{bm}})) \\ & \operatorname{toss}(\mathbf{bm}, r; (p + p')) &= (\operatorname{toss}(\mathbf{bm}, r; p) + \operatorname{toss}(\mathbf{bm}, r; p')) \\ & \operatorname{toss}(\mathbf{bm}, r; (pp')) &= (\operatorname{toss}(\mathbf{bm}, r; p) \operatorname{toss}(\mathbf{bm}, r; p')) \end{aligned}$$

The calculus Soundness Completeness

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The calculus Soundness Completeness

## Tossed formulas

## Let **bm** be a memory cell, $r \in A$ be a constant and p be a probabilistic term.

The formula toss(**bm**,  $r; \eta$ ) is the formula obtained from  $\eta$  by replacing every occurrence of each measure term  $(\int \gamma)$  by  $\tilde{r}(\int \gamma_{\rm tt}^{\rm bm}) + (1 - \tilde{r})(\int \gamma_{\rm ff}^{\rm bm})$ .

 $\begin{aligned} & \operatorname{toss}(\mathbf{bm}, r; \mathrm{fff}) &= \operatorname{fff} \\ & \operatorname{toss}(\mathbf{bm}, r; (p \le p')) &= (\operatorname{toss}(\mathbf{bm}, r; p) \le \operatorname{toss}(\mathbf{bm}, r; p')) \\ & \operatorname{toss}(\mathbf{bm}, r; (\eta \supset \eta')) &= (\operatorname{toss}(\mathbf{bm}, r; \eta) \supset \operatorname{toss}(\mathbf{bm}, r; \eta')) \end{aligned}$ 

The calculus Soundness Completeness

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The calculus Soundness Completeness

## Conditioned terms

#### Let $\gamma$ be classical state formula and ${\it p}$ be a probabilistic term.

The term  $(p/\gamma)$  is the term obtained from p by replacing every occurrence of each measure term  $(\int \gamma')$  by  $(\int (\gamma' \land \gamma))$ .

$$r/\gamma = r$$

$$y/\gamma = y$$

$$(\int \gamma')/\gamma = (\int (\gamma \land \gamma'))$$

$$(p+p')/\gamma = (p/\gamma + p'/\gamma)$$

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The calculus Soundness Completeness

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 $(\eta_1 \uparrow_{\gamma} \eta_2)$  stands for  $((\eta_1/\gamma) \cap (\eta_2/(\neg \gamma)))$ .

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The calculus Soundness Completeness

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The calculus Soundness Completeness

## Axioms

# **[TAUT]** $\vdash \eta$ if $\eta$ is an EPPL theorem $[\int FREE]$ $\vdash \{\kappa\} s \{\kappa\}$ if $\kappa$ is an analytical formula

[SKIP] [ASGR] [ASGB] [TOSS]  $\vdash \{\eta\} \operatorname{skip} \{\eta\}$ 

- $\vdash \{\eta_t^{\mathsf{xm}}\}\,\mathsf{xm} \leftarrow t\,\{\eta\}$
- $Dash \{\eta^{\mathsf{bm}}_\gamma\}\,\mathsf{bm} \leftarrow \gamma\,\{\eta\}$ 
  - $\vdash \{\mathsf{toss}(\mathsf{bm},\eta;r)\} \mathsf{toss}(\mathsf{bm},r) \{\eta\}$

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Axioms

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 $[\textbf{TAUT}] \qquad \vdash \eta \qquad \text{if } \eta \text{ is an EPPL theorem} \\ [\int \textbf{FREE}] \qquad \vdash \{\kappa\} s \{\kappa\} \qquad \text{if } \kappa \text{ is an analytical formula} \end{cases}$ 

[SKIP] [ASGR] [ASGB] [TOSS]

 $\vdash \{\eta\} \text{ skip } \{\eta\}$   $\vdash \{\eta_t^{\text{xm}}\} \text{ xm } \leftarrow t \{\eta\}$   $\vdash \{\eta_{\gamma}^{\text{bm}}\} \text{ bm } \leftarrow \gamma \{\eta\}$   $\vdash \{\text{toss}(\text{bm}, \eta; r)\} \text{ toss}(\text{bm}, r) \{$ 

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Axioms

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 $\begin{array}{ll} [\textbf{TAUT}] & \vdash \eta & \text{if } \eta \text{ is an EPPL theorem} \\ [\int \textbf{FREE}] & \vdash \{\kappa\} \, s \, \{\kappa\} & \text{if } \kappa \text{ is an analytical formula} \end{array}$ 

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Axioms

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Axioms

The State Logic: EPPL The Programming Language The Hoare Calculus Conclusions

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Axioms

The State Logic: EPPL The Programming Language The Hoare Calculus Conclusions

The calculus Soundness Completeness

 $\begin{array}{ll} [\textbf{TAUT}] & \vdash \eta & \text{if } \eta \text{ is an EPPL theorem} \\ [\int \textbf{FREE}] & \vdash \{\kappa\} \, s \, \{\kappa\} & \text{if } \kappa \text{ is an analytical formula} \end{array}$ 

 $\begin{array}{ll} [\mathsf{SKIP}] & \vdash \{\eta\} \operatorname{skip} \{\eta\} \\ [\mathsf{ASGR}] & \vdash \{\eta_t^{\mathsf{xm}}\} \operatorname{xm} \leftarrow t \{\eta\} \\ [\mathsf{ASGB}] & \vdash \{\eta_\gamma^{\mathsf{bm}}\} \operatorname{bm} \leftarrow \gamma \{\eta\} \\ [\mathsf{TOSS}] & \vdash \{\operatorname{toss}(\mathsf{bm},\eta;r)\} \operatorname{toss}(\mathsf{bm},r) \{\eta\} \end{array}$ 

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The calculus Soundness Completeness

### Inference rules

## $[SEQ] \qquad \{\eta_0\} \, s_1 \, \{\eta_1\}, \{\eta_1\} \, s_2 \, \{\eta_2\} \vdash \{\eta_0\} \, s_1; s_2 \, \{\eta_2\}$

 $[IF] \qquad \{\eta_1\} \, s_1 \, \{y_1 = (\int \gamma_0)\}, \{\eta_2\} \, s_2 \, \{y_2 = (\int \gamma_0)\} \\ \vdash \{\eta_1 \, \curlyvee_\gamma \, \eta_2\} \text{if } \gamma \text{ then } s_1 \text{ else } s_2 \{y_1 + y_2 = (\int \gamma_0)\}$ 

 $[\textbf{ELIMV}] \qquad \{\eta_1 \cap (y = p)\} s \{\eta_2\} \vdash \{\eta_1_p^y\} s \{\eta_2\}$ y does not occur in p or  $\eta_2$ 

 $\begin{bmatrix} \text{CONS} \end{bmatrix} \quad \eta_0 \supset \eta_1, \{\eta_1\} \ s \ \{\eta_2\}, \eta_2 \supset \eta_3 \vdash \{\eta_0\} \ s \ \{\eta_3\} \\ \begin{bmatrix} \text{OR} \end{bmatrix} \quad \{\eta_0\} \ s \ \{\eta_2\}, \{\eta_1\} \ s \ \{\eta_2\} \vdash \{\eta_0 \cup \eta_1\} \ s \ \{\eta_2\} \\ \begin{bmatrix} \text{AND} \end{bmatrix} \quad \{\eta_0\} \ s \ \{\eta_1\}, \{\eta_0\} \ s \ \{\eta_2\} \vdash \{\eta_0\} \ s \ \{\eta_1 \bigcap \eta_2\} \\ \end{bmatrix}$ 

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The calculus Soundness Completeness

### Substitution Lemma for classical valuations

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For any valuation  $v \in V$ , any classical state formula  $\gamma$ , any memory cell m (**xm** or **bm**) and term e of the same type,

 $v^m_{\llbracket e \rrbracket_v} \Vdash_{\mathsf{c}} \gamma \text{ iff } v \Vdash_{\mathsf{c}} \gamma^m_e.$ 

#### Proof.

Induction on the structure of  $\gamma$ .

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The calculus Soundness Completeness

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The calculus Soundness Completeness

# Substitution Lemma for assignment

#### Lemma

Let  $(\mathcal{K}, \mu)$  be a generalized probabilistic structure and  $\rho$  be a  $\mathcal{K}$ -assignment. Given a memory cell m and a term e of the same type, let  $\mu' = \mu \circ (\delta_e^m)^{-1}$ . Then

 $\llbracket (\int \gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu')} = \llbracket (\int \gamma^m_e) \rrbracket^{\rho}_{(\mathcal{K},\mu)}$ 

for any classical state formula  $\gamma$ . Furthermore, for any probabilistic term p

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The calculus Soundness Completeness

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$$(\delta_e^m)^{-1}(|\gamma|_{\mathcal{V}}) = |\gamma_e^m|_{\mathcal{V}}$$
 and hence  $\mu((\delta_e^m)^{-1}(|\gamma|_{\mathcal{V}})) = \mu(|\gamma_e^m|_{\mathcal{V}}).$ 

### Therefore, by definition

$$\llbracket (\int \gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu')} = \mu \circ (\delta^m_e)^{-1} (|\gamma|_{\mathcal{V}}) = \mu (|\gamma^m_e|_{\mathcal{V}}) = \llbracket (\int \gamma^m_e) \rrbracket^{\rho}_{(\mathcal{K},\mu)}$$

The result is extended to probabilistic terms and formulas by induction.

#### Corollary

Axioms ASGB and ASGR are sound.

The calculus Soundness Completeness

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The calculus Soundness Completeness

## Substitution Lemma for probabilistic tosses

#### Lemma

Let  $(K, \mu)$  be a generalized probabilistic structure,  $\rho$  be a  $\mathcal{K}$ -assignment,  $r \in \mathcal{A}$  be a constant and  $\mu' = \tilde{r}\mu \circ (\delta_{tt}^{bm})^{-1} + (1 - \tilde{r})\mu \circ (\delta_{tt}^{bm})^{-1}.$ 

For any classical state formula  $\gamma$ ,

 $\llbracket (\int \gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu')} = \widetilde{r} \llbracket (\int \gamma^{\mathsf{bm}}_{\mathfrak{t}}) \rrbracket^{\rho}_{(\mathcal{K},\mu)} + (1 - \widetilde{r}) \llbracket (\int \gamma^{\mathsf{bm}}_{\mathrm{ff}}) \rrbracket^{\rho}_{(\mathcal{K},\mu)}$ 

Furthermore, for any probabilistic term p,

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 $(K, \mu')_{0} \Vdash n \text{ iff } (K, \mu)_{0} \Vdash \text{toss}(\mathbf{bm}, r; n).$ 

The calculus Soundness Completeness

# Substitution Lemma for probabilistic tosses

### Proof.

Let  $\mu_1 = \mu \circ (\delta_{tt}^{bm})^{-1}$  and  $\mu_2 = \mu \circ (\delta_{ff}^{bm})^{-1}$ . Then

 $\llbracket (f\gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu')} = \widetilde{r} \llbracket (f\gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu_1)} + (1-\widetilde{r}) \llbracket (f\gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu_2)}$ 

by definition. Also

 $\llbracket (f\gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu_1)} = \llbracket (f\gamma^{\mathsf{bm}}_{\mathtt{t}}) \rrbracket^{\rho}_{(\mathcal{K},\mu)} \text{ and } \llbracket (f\gamma) \rrbracket^{\rho}_{(\mathcal{K},\mu_2)} = \llbracket (f\gamma^{\mathsf{bm}}_{\mathtt{ff}}) \rrbracket^{\rho}_{(\mathcal{K},\mu)}$ 

The claim for probabilistic terms and probabilistic formulas then follows by induction.

#### Corollary

Axiom TOSS is sound.

The calculus Soundness Completeness

# Substitution Lemma for probabilistic tosses

### Proof.

Let 
$$\mu_1 = \mu \circ (\delta_{\text{tt}}^{\text{bm}})^{-1}$$
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The calculus Soundness Completeness

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The calculus Soundness Completeness

# Substitution Lemma for probabilistic tosses

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The calculus Soundness Completeness

# Soundness of $\int FREE$

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For any statement s, any analytical formula  $\kappa$ , any generalized state  $(\mathcal{K}, \mu)$  and  $\mathcal{K}$  assignment  $\rho$ ,

### $(\llbracket s \rrbracket(\mathcal{K},\mu))\rho \Vdash \kappa \text{ iff } (\mathcal{K},\mu)\rho \Vdash \kappa.$

#### Proof.

The interpretation of analytical formulas depends only on  $ho_2$ 

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The calculus Soundness Completeness

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#### Lemma

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The calculus Soundness Completeness

## Soundness of $\ensuremath{\mathsf{IF}}$

#### \_emma

For any generalized state  $(\mathcal{K}, \mu)$ ,  $\mathcal{K}$ -assignment  $\rho$  and classical state formulas  $\gamma$  and  $\gamma'$ ,

 $\llbracket (\int \gamma') / \gamma \rrbracket^{\rho}_{(\mathcal{K},\mu)} = \llbracket (\int \gamma') \rrbracket^{\rho}_{(\mathcal{K},\mu_{\gamma})}$ 

Furthermore, for any probability term p,

$$\llbracket p/\gamma \rrbracket^{\rho}_{(\mathcal{K},\mu)} = \llbracket p \rrbracket^{\rho}_{(\mathcal{K},\mu_{\gamma})},$$

and, for any probabilistic formula  $\eta$ ,

 $(\mathcal{K},\mu)\rho \Vdash \eta/\gamma \text{ iff } (\mathcal{K},\mu_{\gamma})\rho \Vdash \eta.$ 

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The calculus Soundness Completeness

## Soundness of $\ensuremath{\mathsf{IF}}$

#### Lemma

For any generalized state ( $\mathcal{K}$ ,  $\mu$ ),  $\mathcal{K}$ -assignment  $\rho$  and classical state formulas  $\gamma$  and  $\gamma'$ ,

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Furthermore, for any probability term p,

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The calculus Soundness Completeness

## Soundness of $\ensuremath{\mathsf{IF}}$

#### Lemma

For any generalized state ( $\mathcal{K}$ ,  $\mu$ ),  $\mathcal{K}$ -assignment  $\rho$  and classical state formulas  $\gamma$  and  $\gamma'$ ,

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The calculus Soundness Completeness

## Soundness of IF

### Proof.

By definition,

$$\llbracket (\int \gamma') \rrbracket_{(\mathcal{K},\mu_{\gamma})}^{\rho} = \mu_{\gamma}(|\gamma'|_{\mathcal{V}}) = \mu(|\gamma'|_{\mathcal{V}} \cap |\gamma|_{\mathcal{V}}) = \mu(|\gamma' \wedge \gamma|_{\mathcal{V}}) = \\ \llbracket (\int \gamma')/\gamma \rrbracket_{(\mathcal{K},\mu)}^{\rho}.$$

The claims for probabilistic terms and formulas follow by induction.
The calculus Soundness Completeness

# Soundness of IF

## Proof.

By definition,

$$\llbracket (\int \gamma') \rrbracket^{\rho}_{(\mathcal{K},\mu_{\gamma})} = \mu_{\gamma} (|\gamma'|_{\mathcal{V}}) = \mu(|\gamma'|_{\mathcal{V}} \cap |\gamma|_{\mathcal{V}}) = \mu(|\gamma' \wedge \gamma|_{\mathcal{V}}) = \\ \llbracket (\int \gamma') / \gamma \rrbracket^{\rho}_{(\mathcal{K},\mu)}.$$

The claims for probabilistic terms and formulas follow by induction.

The calculus Soundness Completeness

# Soundness of IF

### Corollary

Given probabilistic state formulas  $\eta_1$  and  $\eta_2$ , programs  $s_1$  and  $s_2$ , variables  $y_1 \in Y$  and  $y_2 \in Y$  and a classical state formula  $\gamma$ ,

 $\vDash_h \{\eta_1\} s_1 \{y_1 = (\int \gamma)\} \text{ and } \vDash_h \{\eta_2\} s_2 \{y_2 = (\int \gamma)\}$ 

iff, for any classical state formula  $\gamma_0$ ,

 $\vDash_h \{\eta_1 \curlyvee_{\gamma_0} \eta_2\} \text{ if } \gamma_0 \text{ then } s_1 \text{ else } s_2 \{y_1 + y_2 = (\int \gamma)\}.$ 

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The calculus Soundness Completeness

# Soundness of $\ensuremath{\mathsf{IF}}$

## Corollary

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$$\vDash_{h} \{\eta_{1}\} s_{1} \{y_{1} = (\int \gamma)\} \text{ and } \vDash_{h} \{\eta_{2}\} s_{2} \{y_{2} = (\int \gamma)\}$$

iff, for any classical state formula  $\gamma_0$ ,

 $\vDash_h \{\eta_1 \curlyvee_{\gamma_0} \eta_2\} \text{ if } \gamma_0 \text{ then } s_1 \text{ else } s_2 \{y_1 + y_2 = (\int \gamma)\}.$ 

The calculus Soundness Completeness

# Soundness of IF

## Proof.

Suppose that  $(\mathcal{K},\mu)\rho \Vdash \eta_1 \Upsilon_{\gamma_0} \eta_2$ . Then  $(\mathcal{K},\mu)\rho \Vdash \eta_1/\gamma_0$  and  $(\mathcal{K},\mu)\rho \Vdash \eta_2/(\neg \gamma_0)$ . Thus,  $(\mathcal{K},\mu_{\gamma_0})\rho \Vdash \eta_1$  and  $(\mathcal{K},\mu_{(\neg \gamma_0)})\rho \Vdash \eta_2$ . Let  $(\mathcal{K},\mu_1) = [s_1](\mathcal{K},\mu_{\gamma_0})$ ,  $(\mathcal{K},\mu_2) = [s_2](\mathcal{K},\mu_{(\neg \gamma_0)})$  and  $\mu' = \mu_1 + \mu_2$ . Since  $\ln_n(\eta_1) = [\eta_1 = (\eta_1)$  and  $(\mathcal{K},\mu_n)\rho \Vdash \eta_1$  it follows that  $(\mathcal{K},\mu_1) \Vdash \eta_2 = (\eta_1) = (\eta_1)$ . Similarly,  $\rho(\gamma_2) = \mu_2((\gamma_1))$ . Hence,  $\mu'((\gamma_1)\mu) = \mu_1((\gamma_1)\mu) + \mu_2((\gamma_1)\mu) = \rho(\gamma_1) + \rho(\gamma_2) = \rho(\gamma_1 + \gamma_2)$  and

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The calculus Soundness Completeness

# Soundness of $\ensuremath{\mathsf{IF}}$

## Proof.

Suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_1 \Upsilon_{\gamma_0} \eta_2$ . Then  $(\mathcal{K}, \mu)\rho \Vdash \eta_1/\gamma_0$  and  $(\mathcal{K}, \mu)\rho \Vdash \eta_2/(\neg \gamma_0)$ . Thus,  $(\mathcal{K}, \mu_{\gamma_0})\rho \Vdash \eta_1$  and  $(\mathcal{K}, \mu_{(\neg \gamma_0)})\rho \Vdash \eta_2$ . Let  $(\mathcal{K}, \mu_1) = \llbracket s_1 \rrbracket (\mathcal{K}, \mu_{\gamma_0}), (\mathcal{K}, \mu_2) = \llbracket s_2 \rrbracket (\mathcal{K}, \mu_{(\neg \gamma_0)})$  and  $\mu' = \mu_1 + \mu_2$ . Since  $\Vdash_h \{\eta_1\} s_1 \{y_1 = (\int \gamma)\}$  and  $(\mathcal{K}, \mu_{\gamma_0})\rho \Vdash \eta_1$  it follows that  $(\mathcal{K}, \mu_1) \Vdash_h y_1 = (\int \gamma)$ . Thus, by definition  $\mu(\gamma) = \mu_1(\gamma_1)$  and  $\mu' = \mu_1 + \mu_2$ .

The calculus Soundness Completeness

# Soundness of IF

### Proof.

Suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_1 \Upsilon_{\gamma_0} \eta_2$ . Then  $(\mathcal{K}, \mu)\rho \Vdash \eta_1/\gamma_0$  and  $(\mathcal{K}, \mu)\rho \Vdash \eta_2/(\neg \gamma_0)$ . Thus,  $(\mathcal{K}, \mu_{\gamma_0})\rho \Vdash \eta_1$  and  $(\mathcal{K}, \mu_{(\neg \gamma_0)})\rho \Vdash \eta_2$ . Let  $(\mathcal{K}, \mu_1) = \llbracket s_1 \rrbracket (\mathcal{K}, \mu_{\gamma_0}), (\mathcal{K}, \mu_2) = \llbracket s_2 \rrbracket (\mathcal{K}, \mu_{(\neg \gamma_0)})$  and  $\mu' = \mu_1 + \mu_2$ . Since  $\Vdash_h \{\eta_1\} s_1 \{y_1 = (\int \gamma)\}$  and  $(\mathcal{K}, \mu_{\gamma_0})\rho \Vdash \eta_1$ , it follows that  $(\mathcal{K}, \mu_1) \Vdash_h y_1 = (\int \gamma)$ . Thus, by definition  $\rho(y_1) = \mu_1(|\gamma|\nu)$ .

The calculus Soundness Completeness

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The calculus Soundness Completeness

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### Proof.

Suppose that  $(\mathcal{K},\mu)\rho \Vdash \eta_1 \curlyvee_{\gamma_0} \eta_2$ . Then  $(\mathcal{K},\mu)\rho \Vdash \eta_1/\gamma_0$  and  $(\mathcal{K},\mu)\rho \Vdash \eta_2/(\neg \gamma_0)$ . Thus,  $(\mathcal{K},\mu_{\gamma_0})\rho \Vdash \eta_1$  and  $(\mathcal{K},\mu_{(\neg \gamma_0)})\rho \Vdash \eta_2$ . Let  $(\mathcal{K}, \mu_1) = [s_1](\mathcal{K}, \mu_{\gamma_0}), (\mathcal{K}, \mu_2) = [s_2](\mathcal{K}, \mu_{(\neg \gamma_0)})$  and  $\mu' = \mu_1 + \mu_2.$ Since  $\Vdash_h \{\eta_1\} s_1 \{y_1 = (\lceil \gamma)\}$  and  $(\mathcal{K}, \mu_{\gamma_0}) \rho \Vdash \eta_1$ , it follows that  $(\mathcal{K}, \mu_1) \Vdash_h \gamma_1 = (\int \gamma)$ . Thus, by definition  $\rho(\gamma_1) = \mu_1(|\gamma|_{\mathcal{V}})$ .

The calculus Soundness Completeness

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### Proof.

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The calculus Soundness Completeness

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 $\mu'(|\gamma|_{\mathcal{V}}) = \mu_1(|\gamma|_{\mathcal{V}}) + \mu_2(|\gamma|_{\mathcal{V}}) = \rho(y_1) + \rho(y_2) = \rho(y_1 + y_2) \text{ and} \\ (\mathcal{K}, \mu')\rho \Vdash (y_1 + y_2 = (f\gamma)) \text{ as required.}$ 

The calculus Soundness Completeness

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### Proof.

Suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_1 \curlyvee_{\gamma_0} \eta_2$ . Then  $(\mathcal{K}, \mu)\rho \Vdash \eta_1/\gamma_0$  and  $(\mathcal{K}, \mu)\rho \Vdash \eta_2/(\neg \gamma_0)$ . Thus,  $(\mathcal{K}, \mu_{\gamma_0})\rho \Vdash \eta_1$  and  $(\mathcal{K}, \mu_{(\neg \gamma_0)})\rho \Vdash \eta_2$ . Let  $(\mathcal{K}, \mu_1) = \llbracket s_1 \rrbracket (\mathcal{K}, \mu_{\gamma_0})$ ,  $(\mathcal{K}, \mu_2) = \llbracket s_2 \rrbracket (\mathcal{K}, \mu_{(\neg \gamma_0)})$  and  $\mu' = \mu_1 + \mu_2$ . Since  $\Vdash_h \{\eta_1\} s_1 \{y_1 = (\int \gamma)\}$  and  $(\mathcal{K}, \mu_{\gamma_0})\rho \Vdash \eta_1$ , it follows that  $(\mathcal{K}, \mu_1) \Vdash_h y_1 = (\int \gamma)$ . Thus, by definition  $\rho(y_1) = \mu_1(|\gamma|_{\mathcal{V}})$ . Similarly,  $\rho(y_2) = \mu_2(|\gamma|_{\mathcal{V}})$ . Hence,  $\mu'(|\gamma|_{\mathcal{V}}) = \mu_1(|\gamma|_{\mathcal{V}}) + \mu_2(|\gamma|_{\mathcal{V}}) = \rho(y_1) + \rho(y_2) = \rho(y_1 + y_2)$  and  $(\mathcal{K}, \mu')\rho \Vdash (y_1 + y_2 = (\int \gamma))$  as required.  $\Box$ 

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The calculus Soundness Completeness

# Soundness of **ELIMV**

#### Lemma

Let 
$$k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho}$$
 and  $\rho_1 = \rho_k^y$ . Then:

• for any probabilistic term  $p_0$ ,  $\llbracket p_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \llbracket p_0 P_p \rrbracket_{(\mathcal{K},\mu)}^{\rho};$ 

• for any probabilistic formula  $\eta$ ,  $(\mathcal{K},\mu)\rho_1 \Vdash \eta$  iff  $(\mathcal{K},\mu)\rho \Vdash \eta_p^{y}$ .

#### Proof.

Let  $p_0$  be a variable  $y_0$ . If  $y_0$  is y, then  $\llbracket y \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_p^y \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . Otherwise,  $\llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \rho_1(y_0) = \rho(y_0) = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\gamma}$ . The rest follows by induction.

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The calculus Soundness Completeness

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Let 
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 and  $\rho_1 = \rho_k^y$ . Then:

• for any probabilistic term  $p_0$ ,  $\llbracket p_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \llbracket p_0 \lor_P \rrbracket_{(\mathcal{K},\mu)}^{\rho};$ 

• for any probabilistic formula  $\eta$ ,  $(\mathcal{K}, \mu)\rho_1 \Vdash \eta$  iff  $(\mathcal{K}, \mu)\rho \Vdash \eta_p^y$ .

#### Proof.

Let  $p_0$  be a variable  $y_0$ . If  $y_0$  is y, then  $\llbracket y \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_p^{\gamma} \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . Otherwise,  $\llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \rho_1(y_0) = \rho(y_0) = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\gamma}$ . The rest follows by induction.

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The calculus Soundness Completeness

# Soundness of **ELIMV**

### Lemma

Let 
$$k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{
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- for any probabilistic term  $p_0$ ,  $[\![p_0]\!]_{(\mathcal{K},\mu)}^{\rho_1} = [\![p_0]\!]_{(\mathcal{K},\mu)}^{\varphi};$
- for any probabilistic formula η, (K, μ)ρ<sub>1</sub> ⊢ η iff (K, μ)ρ ⊢ η<sup>y</sup><sub>p</sub>.

#### Proof.

Let  $p_0$  be a variable  $y_0$ . If  $y_0$  is y, then  $\llbracket y \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_p^y \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . Otherwise,  $\llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \rho_1(y_0) = \rho(y_0) = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . The rest follows by induction.

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The calculus Soundness Completeness

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- for any probabilistic term  $p_0$ ,  $[\![p_0]\!]_{(\mathcal{K},\mu)}^{\rho_1} = [\![p_0]\!]_{(\mathcal{K},\mu)}^{\varphi};$
- for any probabilistic formula η, (K, μ)ρ<sub>1</sub> ⊢ η iff (K, μ)ρ ⊢ η<sup>y</sup><sub>p</sub>.

### Proof.

### Let $p_0$ be a variable $y_0$ .

If  $y_0$  is y, then  $\llbracket y \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_p^{\gamma} \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . Otherwise,  $\llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \rho_1(y_0) = \rho(y_0) = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\gamma}$ . The rest follows by induction.

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The calculus Soundness Completeness

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### Lemma

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- for any probabilistic formula η, (K, μ)ρ<sub>1</sub> ⊢ η iff (K, μ)ρ ⊢ η<sup>y</sup><sub>p</sub>.

### Proof.

Let  $p_0$  be a variable  $y_0$ . If  $y_0$  is y, then  $\llbracket y \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_p^y \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . Otherwise,  $\llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \rho_1(y_0) = \rho(y_0) = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_{0p} \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . The rest follows by induction.

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The calculus Soundness Completeness

# Soundness of **ELIMV**

### Lemma

Let 
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- for any probabilistic term  $p_0$ ,  $[\![p_0]\!]_{(\mathcal{K},\mu)}^{\rho_1} = [\![p_0]\!]_{(\mathcal{K},\mu)}^{\varphi};$
- for any probabilistic formula η, (K, μ)ρ<sub>1</sub> ⊢ η iff (K, μ)ρ ⊢ η<sup>y</sup><sub>p</sub>.

### Proof.

Let  $p_0$  be a variable  $y_0$ . If  $y_0$  is y, then  $\llbracket y \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_p^y \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . Otherwise,  $\llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \rho_1(y_0) = \rho(y_0) = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . The rest follows by induction.

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The calculus Soundness Completeness

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- for any probabilistic formula η, (K, μ)ρ<sub>1</sub> ⊢ η iff (K, μ)ρ ⊢ η<sup>y</sup><sub>p</sub>.

### Proof.

Let  $p_0$  be a variable  $y_0$ . If  $y_0$  is y, then  $\llbracket y \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = k = \llbracket p \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_p^{\gamma} \rrbracket_{(\mathcal{K},\mu)}^{\rho}$ . Otherwise,  $\llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho_1} = \rho_1(y_0) = \rho(y_0) = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\rho} = \llbracket y_0 \rrbracket_{(\mathcal{K},\mu)}^{\gamma}$ . The rest follows by induction.

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The calculus Soundness Completeness

## Soundness of **ELIMV**

#### Lemma

Given y not occurring in either p or in  $\eta$ ,

if  $\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$  then  $\Vdash_h \{\eta_1^y\} s \{\eta_2\}$ .

### Proof.

Assume that  $\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$  and suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_1^{p}$ . Let  $k = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho}$  and  $\rho_1 = \rho_k^{y}$ . Then  $(\mathcal{K}, \mu)\rho_1 \Vdash \eta_1$  and  $\llbracket y \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = k$ . Also  $\llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = \llbracket p_p^{y} \rrbracket_{(\mathcal{K}, \mu)}^{\rho} = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho} = k$ . Therefore,  $(\mathcal{K}, \mu)\rho_1 \Vdash (y = p)$ . Since  $\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$  and  $\rho_1$  and  $\rho$  differ only in the value assigned to y, which does not occur in  $\eta_2$ ,  $(\llbracket s \rrbracket (\mathcal{K}, \mu))\rho \Vdash \eta_2$ as required.

The calculus Soundness Completeness

# Soundness of **ELIMV**

### Lemma

Given y not occurring in either p or in  $\eta$ ,

if 
$$\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$$
 then  $\Vdash_h \{\eta_1^y\} s \{\eta_2\}$ .

### Proof.

Assume that  $\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$  and suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_1\rho$ . Let  $k = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1}$  and  $\rho_1 = \rho_k^{\gamma}$ . Then  $(\mathcal{K}, \mu)\rho_1 \Vdash \eta_1$  and  $\llbracket y \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = k$ . Also  $\llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = \llbracket p_p^{\gamma} \rrbracket_{(\mathcal{K}, \mu)}^{\rho} = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho} = k$ . Therefore,  $(\mathcal{K}, \mu)\rho_1 \Vdash (y = p)$ . Since  $\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$  and  $\rho_1$  and  $\rho$  differ only in the value assigned to y, which does not occur in  $\eta_2$ ,  $(\llbracket s \rrbracket (\mathcal{K}, \mu))\rho \Vdash \eta_2$ as required.

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## Proof.

Assume that  $\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$  and suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_1^{\gamma_1}$ . Let  $k = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1}$  and  $\rho_1 = \rho_k^{\gamma_1}$ . Then  $(\mathcal{K}, \mu)\rho_1 \Vdash \eta_1$  and  $\llbracket y \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = k$ . Also  $\llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = \llbracket p_p^{\gamma_1} \rrbracket_{(\mathcal{K}, \mu)}^{\rho_2} = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho_2} = k$ . Therefore,  $(\mathcal{K}, \mu)\rho_1 \Vdash (y = p)$ . Since  $\ln_n \lfloor \eta_1 \cap (y = p) \rfloor$  is the land  $\rho_1$  and  $\rho_2 \parallel \rho_1 \parallel \rho_2 \parallel \rho_1 \parallel \rho_1 \parallel \rho_1 \parallel \rho_2 \parallel \rho_2 \parallel \rho_2 \parallel \rho_1 \parallel \rho_1 \parallel \rho_2 \parallel \rho_1 \parallel \rho_2 \parallel \rho_1 \parallel$ 

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Assume that  $\Vdash_h \{\eta_1 \cap (y = p)\} s \{\eta_2\}$  and suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_1^{\mathcal{Y}}\rho$ . Let  $k = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho}$  and  $\rho_1 = \rho_k^{\mathcal{Y}}$ . Then  $(\mathcal{K}, \mu)\rho_1 \Vdash \eta_1$  and  $\llbracket y \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = k$ . Also  $\llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho_1} = \llbracket \rho_p^{\mathcal{Y}} \rrbracket_{(\mathcal{K}, \mu)}^{\rho} = \llbracket p \rrbracket_{(\mathcal{K}, \mu)}^{\rho} = k$ . Therefore,  $(\mathcal{K}, \mu)\rho_1 \Vdash (y = \rho)$ . Since  $\Vdash_h \{\eta_1 \cap (y = \rho)\} s \{\eta_2\}$  and  $\rho_1$  and  $\rho$  differ only in the value assigned to y, which does not occur in  $\eta_2$ .  $(\llbracket s \rrbracket_{(\mathcal{K}, \mu)})\rho \Vdash \eta_2$  as required.

The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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Assume that  $\Vdash_{h} \{\eta_{1} \cap (y = p)\} s \{\eta_{2}\}$  and suppose that  $(\mathcal{K}, \mu)\rho \Vdash \eta_{1}p^{\rho}$ . Let  $k = \llbracket p \rrbracket^{\rho}_{(\mathcal{K}, \mu)}$  and  $\rho_{1} = \rho_{k}^{y}$ . Then  $(\mathcal{K}, \mu)\rho_{1} \Vdash \eta_{1}$  and  $\llbracket y \rrbracket^{\rho_{1}}_{(\mathcal{K}, \mu)} = k$ . Also  $\llbracket p \rrbracket^{\rho_{1}}_{(\mathcal{K}, \mu)} = \llbracket p_{p}^{y} \rrbracket^{\rho}_{(\mathcal{K}, \mu)} = \llbracket p \rrbracket^{\rho}_{(\mathcal{K}, \mu)} = k$ . Therefore,  $(\mathcal{K}, \mu)\rho_{1} \Vdash (y = p)$ . Since  $\Vdash_{h} \{\eta_{1} \cap (y = p)\} s \{\eta_{2}\}$  and  $\rho_{1}$  and  $\rho$  differ only in the value assigned to y, which does not occur in  $\eta_{2}$ ,  $(\llbracket s \rrbracket (\mathcal{K}, \mu))\rho \Vdash \eta_{2}$ as required.

The calculus Soundness Completeness

## Soundness of the calculus

#### Theorem

*If*  $\vdash \Psi$  *then*  $\models_h \Psi$ .

### Proof.

By induction on the length of the derivation of  $\vdash \Psi$  using the previous lemmas.

The calculus Soundness Completeness

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The calculus Soundness Completeness

## Preterms

pt(skip, p) = p  $pt(bm \leftarrow \gamma, p) = p_{\gamma}^{bm}$   $pt(xm \leftarrow t, p) = p_{t}^{xm}$  pt(toss(bm, r), p) = toss(bm, r; p)  $pt(s_{1}; s_{2}, p) = pt(s_{1}, pt(s_{2}, p))$ 

The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

## Preterms

 $pt(if \gamma then s_1 else s_2, r) = r$   $pt(if \gamma then s_1 else s_2, y) = y$   $pt(if \gamma then s_1 else s_2, (\int \gamma_0)) = (pt(s_1, (\int \gamma_0))/\gamma + pt(s_2, (\int \gamma_0))/(\neg \gamma))$   $pt(if \gamma then s_1 else s_2, (p_1 + p_2)) = (pt(if \gamma then s_1 else s_2, p_1) + pt(if \gamma then s_1 else s_2, p_2))$   $pt(if \gamma then s_1 else s_2, (p_1 p_2)) = (pt(if \gamma then s_1 else s_2, p_1) \times pt(if \gamma then s_1 else s_2, p_2))$ 

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The calculus Soundness Completeness

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The calculus Soundness Completeness

## Preterms

 $\begin{array}{rcl} \operatorname{pt}(\operatorname{if} \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ r) &=& r \\ \operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ y) &=& y \\ \operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ (\int \gamma_0)) &=& (\operatorname{pt}(s_1, \ (\int \gamma_0))/(\neg \gamma)) \\ \operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ (p_1 + p_2)) &=& (\operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ p_1) + \\ \operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ (p_1 \ p_2)) &=& (\operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ p_1) + \\ \operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ p_2)) \\ \operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ p_1) \times \\ \operatorname{pt}(\operatorname{if} \ \gamma \ \operatorname{then} \ s_1 \ \operatorname{else} \ s_2, \ p_2)) \end{array}$ 

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The calculus Soundness Completeness

# Properties of preterms

### Lemma

$$\llbracket \mathsf{pt}(s, \, \rho) \rrbracket^{\rho}_{(\mathcal{K}, \mu)} = \llbracket \rho \rrbracket^{\rho}_{\llbracket s \rrbracket(\mathcal{K}, \mu)}.$$

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The calculus Soundness Completeness

## Weakest preconditions

$$\begin{array}{lll} \mathsf{wp}(s,\mathrm{fff}) &=& \mathrm{fff} \\ \mathsf{wp}(s,(p_1 \leq p_2)) &=& (\mathsf{pt}(s,\,p_1) \leq \mathsf{pt}(s,\,p_2)) \\ \mathsf{wp}(s,(\eta_1 \supset \eta_2)) &=& (\mathsf{wp}(s,\eta_1) \supset \mathsf{wp}(s,\eta_2)) \end{array}$$

#### Theorem

 $(\mathcal{K},\mu)\rho \Vdash_h wp(s,\eta) \text{ iff } (\llbracket s \rrbracket (\mathcal{K},\mu))\rho \Vdash_h \eta.$ 

The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

# Weakest preconditions, semantically

### Corollary

$$\vDash_{h} \{\eta'\} s \{\eta\} \text{ iff } \vDash (\eta' \supset wp(s, \eta)).$$

### Proof.

(⇒) Suppose that  $\vDash_h \{\eta'\} s \{\eta\}$  and  $(\mathcal{K}, \mu)\rho \Vdash \eta'$ . Then  $(\llbracket s \rrbracket (\mathcal{K}, \mu))\rho \Vdash \eta$ , hence  $(\mathcal{K}, \mu)\rho \Vdash wp(s, \eta)$ . Therefore  $\vDash (\eta' \supset wp(s, \eta))$ .

( $\Leftarrow$ ) Suppose that  $\vDash (\eta' \supset wp(s, \eta))$  and  $(\mathcal{K}, \mu)\rho \Vdash \eta'$ . Then  $(\mathcal{K}, \mu)\rho \Vdash wp(s, \eta)$  and hence  $(\llbracket s \rrbracket (\mathcal{K}, \mu))\rho \Vdash \eta$ . Therefore  $\vDash_h \{\eta'\} s \{\eta\}$ .

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The calculus Soundness Completeness

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(⇐) Suppose that  $\vDash$  ( $\eta' \supset$  wp( $s, \eta$ )) and ( $\mathcal{K}, \mu$ ) $\rho \Vdash \eta'$ . Then ( $\mathcal{K}, \mu$ ) $\rho \Vdash$  wp( $s, \eta$ ) and hence ( $\llbracket s \rrbracket (\mathcal{K}, \mu)$ ) $\rho \Vdash \eta$ . Therefore =<sub>h</sub> { $\eta'$ } s { $\eta$ }.

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The calculus Soundness Completeness

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(⇒) Suppose that  $\vDash_h \{\eta'\} s \{\eta\}$  and  $(\mathcal{K}, \mu)\rho \Vdash \eta'$ . Then (**[***s***]** $(\mathcal{K}, \mu))\rho \Vdash \eta$ , hence  $(\mathcal{K}, \mu)\rho \Vdash wp(s, \eta)$ . Therefore  $\vDash (\eta' \supset wp(s, \eta))$ .

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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The calculus Soundness Completeness

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$$\vDash_{h} \{\eta'\} s \{\eta\} \text{ iff } \vDash (\eta' \supset wp(s, \eta)).$$

### Proof.

(⇒) Suppose that  $\vDash_h \{\eta'\} s \{\eta\}$  and  $(\mathcal{K}, \mu)\rho \Vdash \eta'$ . Then  $(\llbracket s \rrbracket (\mathcal{K}, \mu))\rho \Vdash \eta$ , hence  $(\mathcal{K}, \mu)\rho \Vdash wp(s, \eta)$ . Therefore  $\vDash (\eta' \supset wp(s, \eta))$ .

( $\Leftarrow$ ) Suppose that  $\vDash$  ( $\eta' \supset wp(s, \eta)$ ) and ( $\mathcal{K}, \mu$ ) $\rho \Vdash \eta'$ . Then ( $\mathcal{K}, \mu$ ) $\rho \Vdash wp(s, \eta)$  and hence ( $\llbracket s \rrbracket (\mathcal{K}, \mu)$ ) $\rho \Vdash \eta$ . Therefore  $\vDash_h \{\eta'\} s \{\eta\}$ .

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The calculus Soundness Completeness

# Weakest preconditions, sintactically

### Lemma

For any probabilistic term p, statement s and variable y,

$$\vdash \{y = \mathsf{pt}(s, p)\} s \{y = p\}.$$

#### Theorem

For any statement s and any conditional-free formula  $\eta$ ,

 $- \{ wp(s, \eta) \} s \{ \eta \}.$ 

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## Completeness and decidability

#### Theorem

Let *s* be a probabilistic sequential program and  $\eta$  be an EPPL formula. If  $\vDash_h \{\eta'\} s \{\eta\}$ , then  $\vdash \{\eta'\} s \{\eta\}$ .

Moreover, the set of theorems of the Hoare calculus is recursive.

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### Proof.

*Completeness.* Suppose that  $\vDash_h \{\eta'\} s \{\eta\}$ . Then  $\vDash (\eta' \supset wp(s, \eta))$ . By completeness of EPPL,  $\vdash (\eta' \supset wp(s, \eta))$ . On the other hand,  $\vdash \{wp(s, \eta)\} s \{\eta\}$ , whence  $\vdash \{\eta'\} s \{\eta\}$  by **CONS**.

Decidability. By soundness and completeness,  $\vdash \{\eta'\} s \{\eta\}$  iff  $\models_h \{\eta'\} s \{\eta\}$ . By completeness of EPPL and the properties of weakest preconditions, it follows that  $\vdash \{\eta'\} s \{\eta\}$  iff  $\vdash (\eta' \supset wp(s, \eta))$ . The decidability is now a consequence of the decidability of EPPL and the fact that  $wp(s, \eta)$  can be computed algorithmically.

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## Achievements

- logic for non-deterministic programs with truth-functional semantics
- sound, complete and decidable state logic
- sound, complete and decidable Hoare calculus

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## Future work

## • unbounded iteration (while)

quantum programming languages

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