sorting networks
the end game

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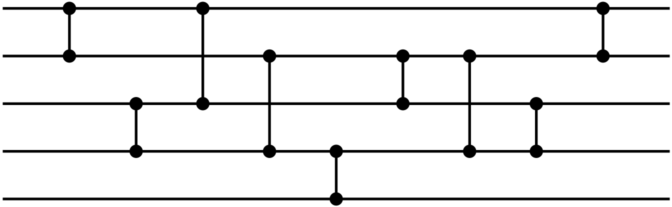
outline

a sorting network



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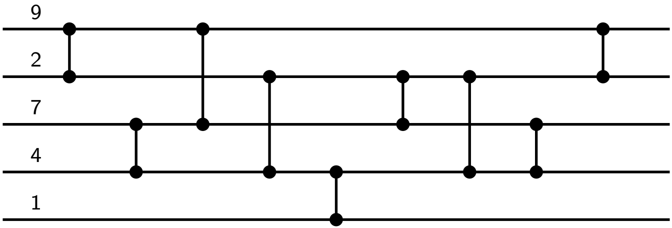

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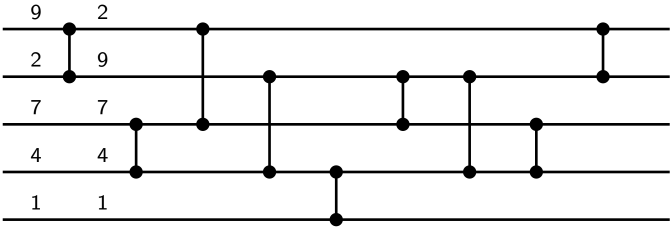

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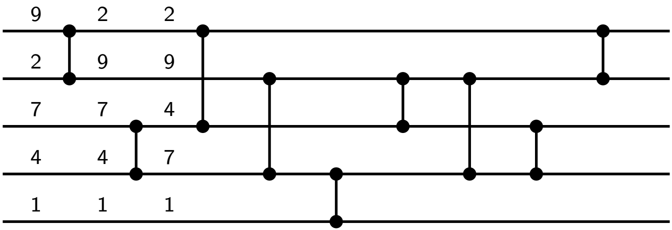

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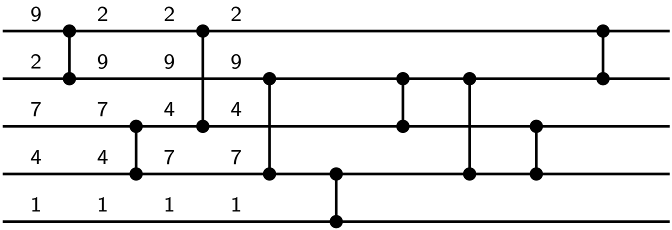

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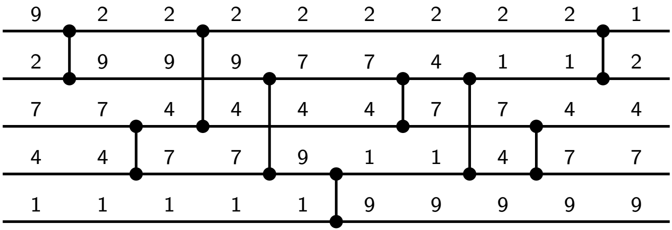

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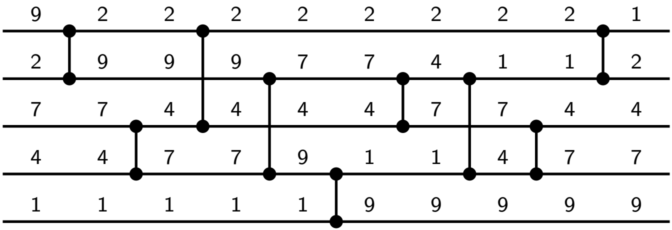

size this net has 5 *channels* and 9 *comparators*

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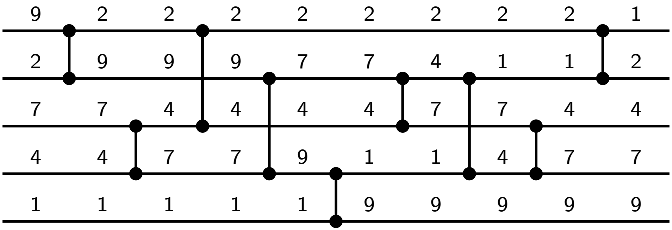

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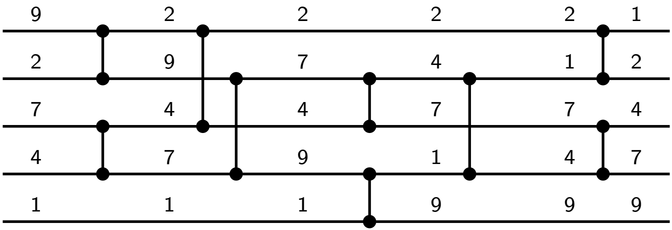

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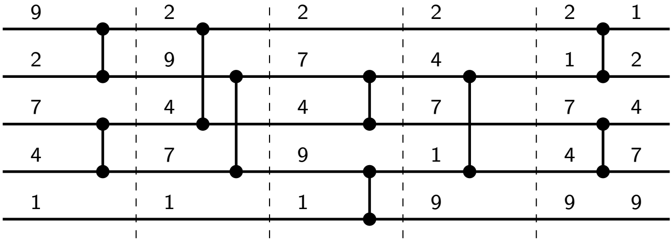

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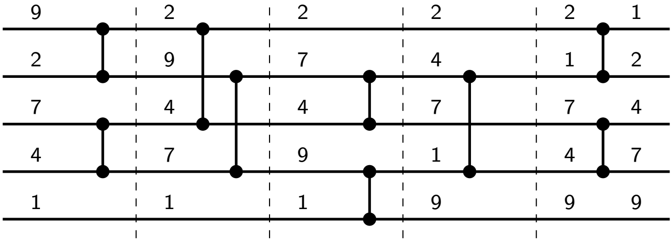

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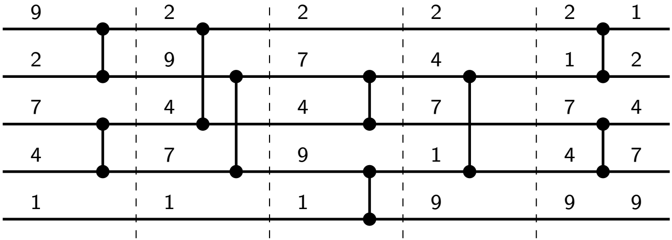

size this net has 5 *channels* and 9 *comparators*

depth this net has 5 *layers*

more info see d.e. knuth, *the art of computer programming*, vol. 3

the optimization problems

the optimal size problem
: what is the minimal number of *comparators* on a sorting network on n channels (s_n)?

the optimal depth problem
: what is the minimal number of *layers* on a sorting network on n channels (t_n)?

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knuth 1973

n	1	2	3	4	5	6	7	8	9	10
t_n	0	1	3	3	5	5	6	6	7 6	7 6

n	11	12	13	14	15	16	17
t_n	8 6	8 6	9 6	9 6	9 6	9 6	11 6

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parberry 1991

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n	11	12	13	14	15	16	17
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bundala & závodný 2013

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ehlers & müller 2014

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an exponential explosion

- upper bounds obtained by concrete examples (1960s)
- lower bounds obtained by mathematical arguments
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bundala & závodný 2013

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however... these techniques do not scale for t_{17}

- sat-solvers cannot handle two-layer prefixes
- too many possibilities for third layer

our proposal

main idea study the properties of the *last* layers of sorting networks

our proposal

main idea study the properties of the *last* layers of sorting networks

- (surprisingly) never done before
- very different problem
- significant drop on sat-solving times
- maybe can handle $n = 17$ (experiment running...)

outline

redundancy

redundant comparator

let $C; (i,j); C'$ be a comparator network
the comparator (i,j) is *redundant* if $x_i \leq x_j$ for all sequences $x_1 \ldots x_n \in \text{outputs}(C)$

lemma

if D and D' only differ in redundant comparators, then D is a sorting network iff D' is a sorting network

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lemma if D and D' only differ in redundant comparators, then D is a sorting network iff D' is a sorting network

goal restrict the search space by disallowing redundant comparators

problem redundancy is a semantic property
$\rightsquigarrow$ not easily encodable in sat

the last layer

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theorem there are $f_{n+1} - 1$ possible last layers in an n-channel sorting network with no redundancy

fibonacci sequence $f_1 = f_2 = 1,\ f_{n+2} = f_{n+1} + f_n$

$\rightsquigarrow$ this reduces the number of possible last layers on 17 channels from 211,799,312 to just 2,583

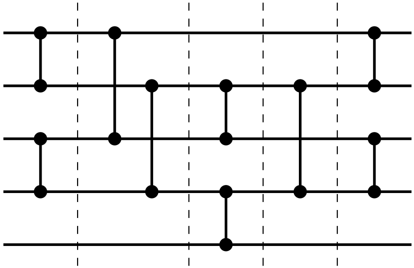
blocks i/ii

k-block a k-block in a sorting network is a set of channels that are connected after layer k

blocks i/ii

k-block

a *k*-block in a sorting network is a set of channels that are connected after layer *k*



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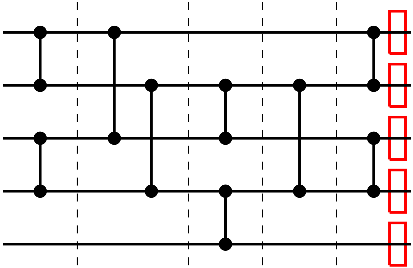

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blocks i/ii

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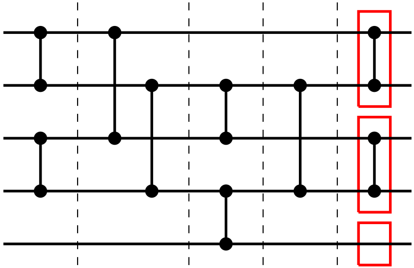

4-blocks

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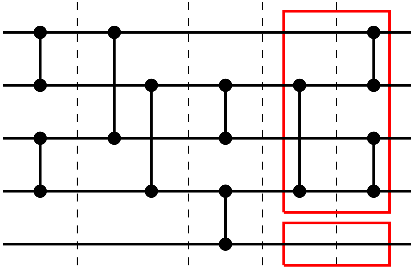

3-blocks

blocks i/ii

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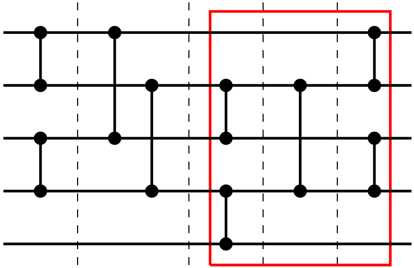

2-blocks

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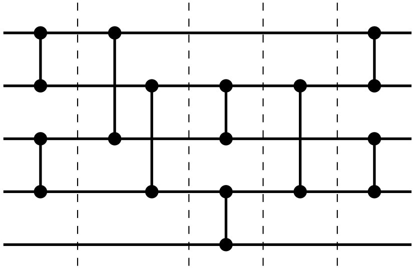

lemma for every input $\bar{x} \in \{0,1\}^n$, there is at most one k-block that receives both 0s and 1s as inputs

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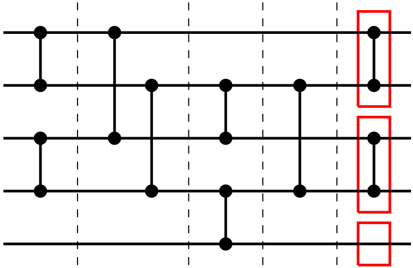

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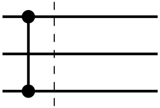

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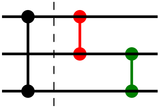

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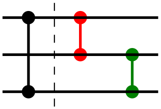

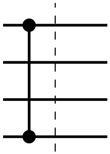

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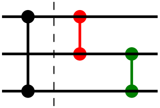

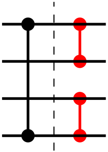

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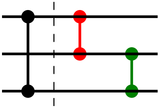

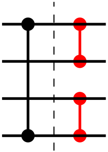

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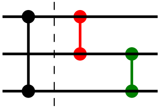

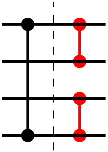

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. . . but it is not enough

outline

revisiting the last layer

new idea

we can reduce the search state even more by *adding* redundant comparators!

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llnf a sorting network is in *last layer normal form* if

- its last layer only contains comparators between adjacent channels
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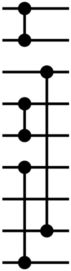

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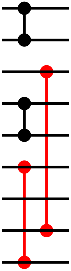

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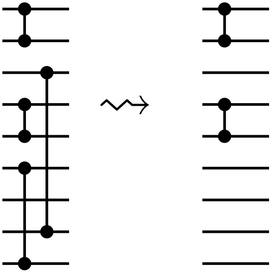

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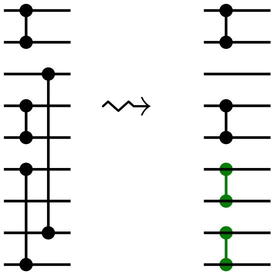

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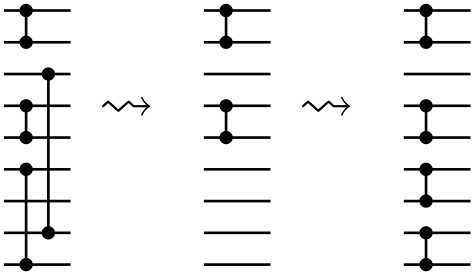

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theorem there are p_{n+5} last layers in llnf on n channels

padovan sequence $p_0 = 1,\ p_1 = p_2 = 0,\ p_{n+3} = p_{n+1} + p_n$

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$\leadsto$ this further reduces the number of possible last layers on 17 channels from 2,583 to only 86

co-saturation i/ii

generalization we can apply the same reasoning to previous layers

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lemma if $i < j$ are two channels unused in layer k of a sorting network belonging to different blocks, then the comparator (i, j) in layer k is redundant

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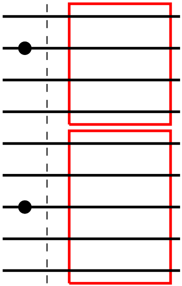

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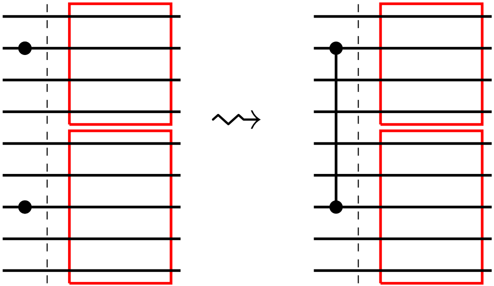

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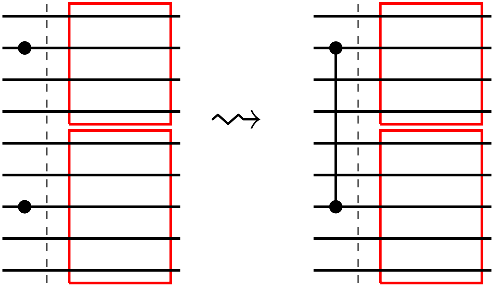

lemma (some stuff about “sliding” comparators)

co-saturation ii/ii

co-saturation

we can characterize the networks resulting from applying these transformations to the two last layers

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co-saturation theorem
if there is a sorting network on n channels, then there is a co-saturated sorting network on n channels with the same depth

co-saturation ii/ii

co-saturation we can characterize the networks resulting from applying these transformations to the two last layers

co-saturation theorem if there is a sorting network on n channels, then there is a co-saturated sorting network on n channels with the same depth

$\rightsquigarrow$ for $n = 17$, there are only 45,664 possibilities for the last two layers of a co-saturated sorting network

practical impact

the good news

we can encode co-saturation in sat

		unrestricted last two layers			
		slowest instance			total
n	#cases	#clauses	#vars	time	time
15	262	278,312	18,217	754.74	**130,551.42**
16	211	453,810	27,007	1,779.14	**156,883.21**

		co-saturated last two layers			
		slowest instance			total
n	#cases	#clauses	#vars	time	time
15	262	335,823	25,209	148.35	**19,029.26**
16	211	314,921	22,901	300.07	**24,604.53**

outline

results

- $6\times$ speedup on optimal depth problem
- similar techniques give $4\times$ speedup on optimal size problem
- can find 10-layer sorting network on 17 channels in one hour
- exact value of t_{17} being computed as we speak

thank you!