a formalized checker for size-optimal sorting networks

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types meeting
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outline

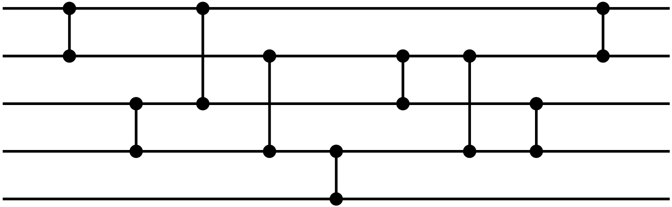
sorting networks in a nutshell

sorting networks, coq style

generate-and-prune

conclusions & future work

a sorting network



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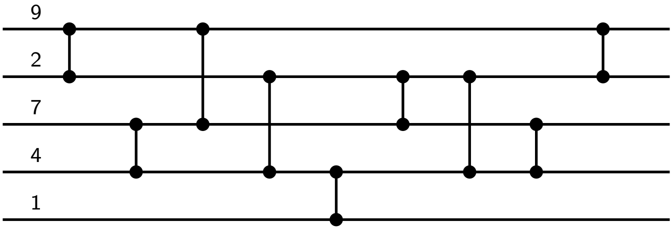

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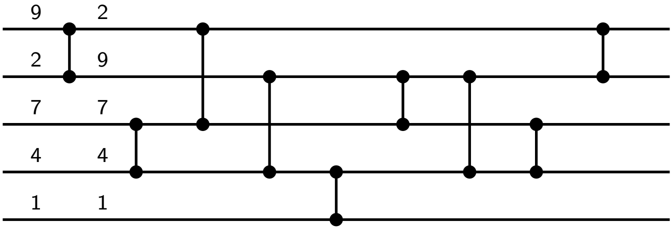

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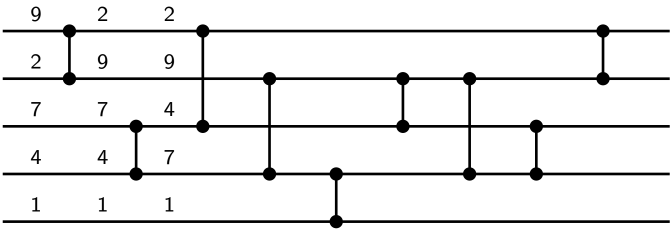

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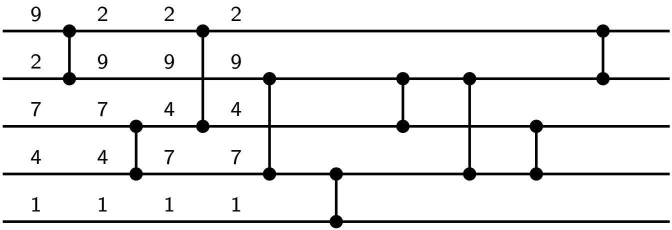

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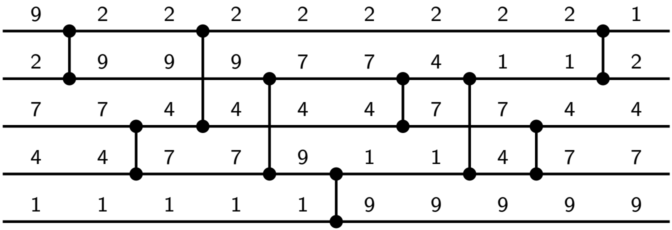

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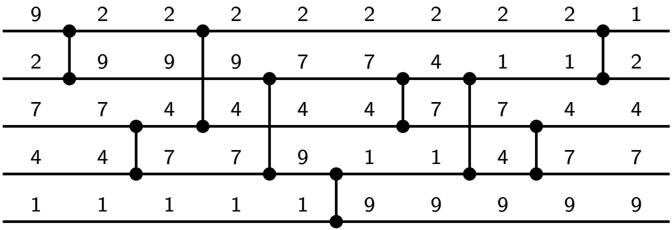

size this net has 5 *channels* and 9 *comparators*

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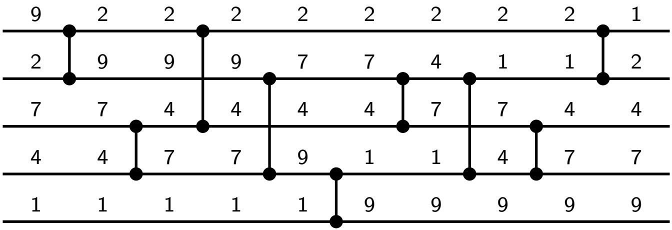

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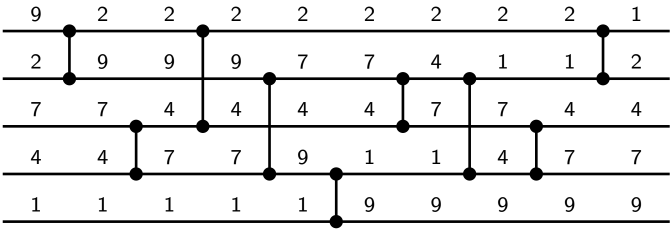

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the optimal size problem what is the minimal number of *comparators* in a sorting network on n channels (s_n)?

history

optimal size

s_n: minimal number of *comparisons* to sort n inputs

knuth 1973

n	1	2	3	4	5	6	7	8	9	10
s_n	0	1	3	5	9	12	16	19	25 23	29 27

n	11	12	13	14	15	16	17
s_n	35 31	39 35	45 39	51 43	56 47	60 51	73 56

- values for $n \leq 4$ from information theory
- values for $n = 5$ and $n = 7$ by exhaustive case analysis

knuth $s_n \geq s_{n-1} + 3$ $\rightsquigarrow$ values for $n = 6, 8$

van voorhis $s_n \geq s_{n-1} + \lg(n)$ $\rightsquigarrow$ other lower bounds

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yours truly 2014

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- generate-and-prune algorithm
- intensive parallel computing
- $\sim$ 16 years of cpu time to compute s_9

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- generate-and-prune algorithm
- intensive parallel computing
- $\sim$ 16 years of cpu time to compute s_9

- but how do we know that these results are correct?

outline

pros and cons

the easy stuff

- (very) constructive theory
- everything is decidable
- many proofs by exhaustive case analysis
- elementary definitions

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main challenges

- all finite domains (channels, inputs, . . .)
- reasoning about permutations (in proofs)
- very informal proofs ("trivial", "exercise", "clearly")

comparator network

sequence of *comparators* (i, j) with $1 \leq i \neq j \leq n$
n is the number of channels

```
Definition comparator : Set := (prod nat nat).
Definition comp_net : Set := list comparator.

Definition comp_channels (n:nat) (c:comparator) :=
  let (i,j) := c in (i<n) /\ (j<n) /\ (i<>j).

Definition channels (n:nat) (C:comp_net) :=
  forall c:comparator, (In c C) -> (comp_channels n c).
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comparator networks

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intuition

$(1, 3), (2, 4)$ is a comparator network on 4 channels, but also on 6 channels

comparator networks

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```

standard

$i < j$ for all $(i, j) \in C$

```
Definition comp_standard (n:nat) (c:comparator) :=
  let (i,j) := c in (i<n) /\ (j<n) /\ (i<j).

Definition standard (n:nat) (C:comp_net) :=
  forall c:comparator, (In c C) -> (comp_standard n c).
```

sorting networks (i/iii)

0/1 lemma (knuth 1973)

C is a sorting network on n channels iff C sorts all inputs in $\{0,1\}^n$

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```
Inductive bin_seq : nat -> Set :=
  | empty : bin_seq 0
  | zero : forall n:nat, bin_seq n -> bin_seq (S n)
  | one : forall n:nat, bin_seq n -> bin_seq (S n).

Fixpoint get n (s:bin_seq n) (i:nat) : nat := ...
Fixpoint set n (s:bin_seq n) (i:nat) (x:nat)
             : (bin_seq n) := ...
```

- similar to `Vector` from the standard library
- definition of `sorted` (property) and `sort` (operation)
- induction principles, exhaustive enumeration
- $\sim$ 70 lemmas in total

sorting networks (ii/iii)

output

$C(\vec{x})$ denotes the *output* of C on $\vec{x} = x_1 \ldots x_n$

```
Fixpoint apply (c:comparator) n (s:bin_seq n) : (bin_seq n) :=
  let (i,j):=c in let x:=(get s i) in let y:=(get s j) in
    match (le_lt_dec x y) with
    | left _ => s
    | right _ => set (set s j x) i y
    end.

Fixpoint full_apply (C:comp_net) n (s:bin_seq n)
  : (bin_seq n) :=
  match C with
  | nil => s
  | cons c C' => full_apply C' _ (apply c s)
  end.
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binary outputs

$\text{outputs}(C) = \{C(\vec{x}) \mid x \in \{0,1\}^n\}$

```
Definition outputs (C:comp_net) (n:nat) : (list (bin_seq n))
  := (map (full_apply C (n:=n)) (all_bin_seqs n)).
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sorting network

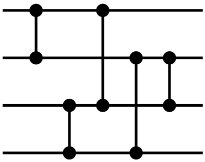
$C(\vec{x})$ is sorted for every input $\vec{x}$

```
Definition sort_net (n:nat) (C:comp_net) :=
  (channels n C) /\
  forall s:bin_seq n, sorted (full_apply C s).

Theorem SN_char : forall C n, channels n C ->
       (forall s, In s (outputs C n) -> sorted s) ->
                 sort_net n C.
```

sorting networks (iii/iii)

sanity check



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  (0[<]1 :: 2[<]3 :: 0[<]2 ::
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Theorem SN4_SN: sort_net 4 SN4.
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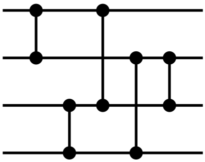

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the bad news

does not scale for 9 channels

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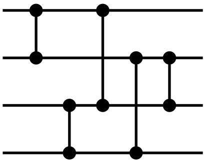

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the good news "C is a sorting network" is decidable

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Lemma SN_dec : forall n C, channels n C ->
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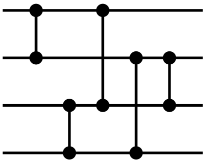

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- program extraction $\leadsto$ haskell program (tests all inputs)
- nearly best possible algorithm (known result)
- short formalization ($\sim$ 35 lemmas)

the key result

output lemma (parberry 1991)

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proof

$$\begin{array}{ccccc} \{0,1\}^n & \xrightarrow{C} & X & & \\ & & \downarrow \pi & & \\ \{0,1\}^n & \xrightarrow{C'} & X' & \xrightarrow{N} & S \end{array}$$

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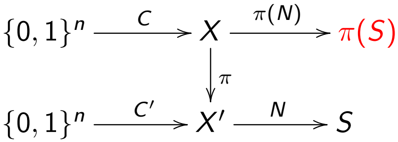

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$\rightsquigarrow$ how do we formalize this?

standardization (i/ii)

standardization take the first non-standard comparator (i, j) and interchange i and j in all subsequent positions; repeat until network is standard

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$\rightsquigarrow$ in our case: need a (simple?) generalization

standardization (ii/ii)

standardization

```
Function standardize (C:comp_net) {measure length C}
  : comp_net := match C with
  | nil => nil
  | cons c C' => let (x,y) := c in
      match (le_lt_dec x y) with
      | left _ => (x[<]y :: standardize C')
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      end
  end.
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```

- not structurally decreasing
- lots of implicit properties
  - preserves size and number of channels
  - preserves standard prefix
  - result is standard
  - idempotent

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Theorem standardization_sort : forall C n,
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$\rightsquigarrow$ requires $\sim$ 60 lemmas about permutations

subsumption

definition $C \preceq_\pi C'$ if $\pi(\text{outputs}(C)) \subseteq \text{outputs}(C')$
$C \preceq C'$ if $C \preceq_\pi C'$ for some permutation π

$\rightsquigarrow$ subsumption is reflexive and transitive

subsumption

definition

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$C \preceq C'$ if $C \preceq_\pi C'$ for some permutation π

```
Variable n:nat.
Variables C C':comp_net.
Variable P:permut.
Variable HP:permutation n P.

Definition subsumption :=
    forall s:bin_seq n, In s (outputs C n) ->
                        In (apply_perm P s) (outputs C' n).

Theorem BZ : standard n C -> subsumption ->
    sort_net n (C'++N) ->
    sort_net n (standardize (C ++ apply_perm_to_net P N)).

Lemma subsumption_dec : {subsumption} + {~subsumption}.
```

outline

the algorithm

init set $R_0^n = \{\emptyset\}$ and $k = 0$

repeat until $k > 1$ and $|R_k^n| = 1$

generate N_{k+1}^n extend each net in R_k^n by one comparator in all possible ways

prune to R_{k+1}^n keep only one element from each minimal equivalence class w.r.t. $\preceq^T$

step increase k

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pruning

- quadratic step
- inner loop searches among all permutations typically fails
- record successful subsumptions

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certified checker using recorded subsumptions as an oracle

- replace pruning cycle by oracle calls
- skeptic approach towards oracle
- use program extraction
- verifies all cases up to s_8, requires $\sim$18 years for s_9...

checker soundness

```
Definition Oracle := list (comp_net * comp_net * (list nat)).

Inductive Answer : Set :=
  | yes : nat -> nat -> Answer
  | no : forall n k:nat, forall R:list comp_net,
         NoDup R ->
        (forall C, In C R -> length C = k) ->
        (forall C, In C R -> standard n C) -> Answer
  | maybe : Answer.

Fixpoint Generate_and_Prune (m n:nat) (O:list Oracle) :
   Answer.

Theorem GP_no : forall m n O R HR0 HR1 HR2,
   Generate_and_Prune m n O = no m n R HR0 HR1 HR2 ->
   forall C, sort_net m C -> length C > n.

Theorem GP_yes : forall m n O k,
   Generate_and_Prune m n O = yes m k ->
  (forall C, sort_net m C -> length C >= k) /\
   exists C, sort_net m C /\ length C = k.
```

an offline oracle

typical approach

- call oracle to solve difficult tasks
- check result
- oracle is online, waiting for the next problem

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in our case

- oracle is pre-computed (offline)
- information from oracle guides algorithm
- potential for optimizations

improving the pruning step

old algorithm while oracle has a next subsumption $C \preceq_\pi C'$

1. check that $C \preceq_\pi C'$
2. check that C, C' are in the current set
3. remove C' from the current set

(laziness performs the last two steps together)

improving the pruning step

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new algorithm

while oracle has a next subsumption $C \preceq_\pi C'$

1. check that $C \preceq_\pi C'$
2. store C
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after: check that all stored networks are in the final set

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requirement

- cannot have subsumption chains, e.g. $C_1 \preceq C_2 \preceq C_3$

pre-processing replace chains by endpoint subsumptions (e.g. $C_1 \preceq C_3$)
computing adequate permutation

$\rightsquigarrow$ don't care how, they will be checked anyway!

improving the pruning step

new algorithm

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optimizations

- provide C's in the order they were generated (replaces quadratic step by linear)
- replace lists by search trees (improves performance)
- extract naturals to native integers (unfortunately necessary, but clearly sound)
- represent comparators as a single number (reduces memory consumption)

philosophical considerations

the good news checker verifies s_9 in around 6 days using “moderate” resources

moderate not-so-new commonplace cpu, 64 gb ram

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more good news (almost) no changes to the formalization

- relatively quick changes (a few hours each)
- mostly require proving that optimized version coincides with original version

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- mostly require proving that optimized version coincides with original version

offline oracles a new methodology?

outline

conclusions & future work

results

- formal verification of exact values of s_n for $n \leq 9$
- new methodology (offline oracles)
- able to deal with $\sim$ 27 gb of proof witnesses
- clean separation between formalization ("mathematics") and optimization of checker ("computer science")

next episodes

- formal proof of van voorhis' $s_n \geq s_{n-1} + \lg(n)$ to obtain s_{10}
- other problems in sorting networks
- application of this method to other search-intensive proofs

thank you!