Cormen 16-1

Question (a)

Optimal substructure: Obvious.

Greedy choice property:

Let $x_1 = 1, x_2 = 5, x_3 = 10, x_4 = 25$.
Let x_ℓ be the denomination of the first coin chosen by the greedy algorithm.

Since the greedy algorithm included x_ℓ in its solution, $n \geq x_\ell$.
Since x_ℓ was the first denomination included in the greedy solution, either $x_\ell = 25$ or $n < x_{\ell+1}$.
Hence, no solution contains denominations larger than x_ℓ.

Let A be an optimal solution, i.e., A is a multiset with elements chosen from $\{x_1, x_2, x_3, x_4\}$.
Assume for the purpose of contradiction that $x_\ell \notin A$. Then, the largest denomination in A is no larger than $x_{\ell-1}$, and $x_\ell \geq 5$.

We will prove that there exists a subset $B \subseteq A$ with $\sum_{x \in B} x = x_\ell$, contradicting that A is optimal.

Case 1: $x_\ell = 25$.
Since A is optimal, A contains at most two copies of 10, because three copies of 10 could be replaced by 25+5.

Case 1a: A contains no copies of 10.
A contains only 5 and 1. Since both 5 and 1 divide 25, and the sum of the elements in A is at least 25, there is a subset of A adding up to 25.

Case 1b: A contains exactly one copy of 10.
$A - \{10\}$ contains only 5 and 1. Since both 5 and 1 divide 15, and the sum of the elements in $A - \{10\}$ is at least 15, there is a subset $B' \subseteq A - \{10\}$ adding up to 15. $B' \cup \{10\}$ is a subset of A adding up to 25.

Case 1c: A contains exactly two copies of 10.
$A - \{10, 10\}$ contains only 5 and 1. Clearly, $A - \{10, 10\}$ has a subset adding up to 5.

Case 2: $x_\ell = 10$.
A contains only 5 and 1. Clearly, there is a subset of A adding up to 10.

Case 3: $x_\ell = 5$.
A contains only 1, so there must be a subset of A adding up to 5.

Question (b)

Since 1 is contained in the set of denominations, there is a solution. We prove that the greedy solution is optimal:

Optimal substructure: Obvious.

Greedy choice property: Let $x_i = c^i$, $0 \leq i \leq k$. The proof uses only the fact that x_{i-1} divides x_i, $1 \leq i \leq k$.

Assume that the first coin chosen by the greedy algorithm has denomination x_ℓ.

Since the greedy algorithm included x_ℓ in its solution, $n \geq x_\ell$.
Since x_ℓ was the *first* denomination included in the greedy solution, $x_\ell = x_k$ or $n < x_{\ell+1}$. Hence, no solution contains denominations larger than x_ℓ.

Let A be an optimal solution, i.e., A is a multiset with elements chosen from $\{x_0, x_1, \ldots, x_\ell\}$. Assume for the purpose of contradiction that $x_\ell \notin A$. Then, the largest denomination in A is no larger than $x_{\ell-1}$, and $x_\ell \geq x_1$.

We prove that there exists a subset $B \subseteq A$ with $\sum_{a \in B} a = x_\ell$, contradicting that A is optimal.

We use the following algorithm in the proof.

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CONSTRUCTB(A,ℓ)
S ← x_ℓ
i ← ℓ
A' ← A, B ← ∅
while S > 0   *J*
    i ← i − 1
    while S ≥ x_i and x_i ∈ A'   *I and I'*
        S ← S − x_i
        A' ← A' − {x_i}
        B ← B ∪ {x_i}
return B
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Claim: CONSTRUCTB(A,ℓ) will return a subset of A summing up to x_ℓ.

Proof: We prove that, as long as the elements in B add up to *less* than x_ℓ (i.e., as long as $S > 0$), there is an element in A' that can be added to B without the elements in B summing up to *more* than x_ℓ. Since $A' = A - B$, this is sufficient to prove the claim.

The inner loop has the invariants $I : x_j$ divides S, $0 \leq j \leq i$ and $I' : \sum_{a \in A'} a \geq S$.

Initialization: Initially, $S = x_{i+1}$, and x_j divides x_{j+1}, $0 \leq j \leq i$. This proves I.

Initially, $S = x_\ell \leq n$ and $A' = A$. This proves I'.

Maintenance: Assume that the invariants were true in the previous iteration. Since then, either S has been decremented by x_i, or i has been decremented by 1. In both cases, I will still be true.

If i has been decremented, S and A' are unchanged. Otherwise, x_i has been subtracted from S and removed from A'. This proves that I' is maintained too.

The outer loop has the invariant $J : S = 0$ or $a \leq x_{i-1}$, for all $a \in A'$.

Initialization: Initially, $i = \ell$, and the largest denomination in $A' = A$ is no more than $x_{\ell-1}$.

Maintenance: Assume that the invariant was true in the previous iteration, i.e., assume that $a \leq x_i$, for all $a \in A'$. Then we just need to prove that $S = 0$ or $x_i \notin A'$.

The inner loop terminates only when $S < x_i$ or $x_i \notin A'$.

By I, x_i divides S. Thus, $S < x_i \Leftrightarrow S = 0$.

As long as $S > 0$, I' implies $A' \neq \emptyset$ and I combined with J imply that all elements in A' divide S. Hence, any element in A' is smaller than or equal to S and hence can safely be added to B.