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DM559/DM545 – Linear and integer programming

Sheet 8, Spring 2018 [pdf format]

Exercise 1*

Manpower Planning. Given a set of workers and the need to cover a set of 15 working hours per day with a, possibly different, number of required persons as staff at each hour, decide the staff at each hour taking into consideration that each person works in shifts that cover 7 hours and hence a person starting in hour i contributes to the workload in hours $i, \ldots, i+6$ (e.g., a person starting in hour 3 contributes to the workload in hours 3,4,5,6,7,8,9).
Formulate the problem to determine the number of people required to cover the workload in mathematical programming terms.

Exercise 2*

Give two MILP formulations for the chromatic number problem: given a graph find a coloring (i.e., a mapping from the natural numbers to the set of vertices) such that every pair of vertices connected by an edge receives different colors and the number of colors used is the least possible.

Exercise 3*

Formulate the Minimum Spanning Tree problem as a mathematical programming problem.

Exercise 4*

Consider the polyhedron $P \subseteq R^2$ described by the following inequalities and depicted in Figure 1:

$$\begin{array}{rcrcr}
2x & - & y & \leq & 4 \\
2x & + & 3y & \leq & 12 \\
 & & y & \leq & 3 \\
3x & + & 2y & \geq & 6 \\
x & & & \leq & 3 \\
3/2x & + & y & \leq & 45/8 \\
2x & + & 3y & \leq & 10
\end{array}$$

Which inequality is a face? Which is a facet? Which is redundant?

Exercise 5*

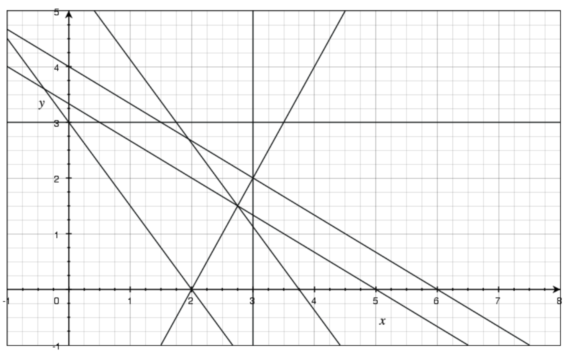

Figure 1: The polyhedron of exercise 4.

Given the following LP problem:

$$\begin{aligned} \min\ & 2x_1 + x_2 + 6x_3 - x_4 \\ & 3x_1 + x_2 - x_5 = 2 \\ & x_2 + 4x_3 = 4 \\ & x_3 + 4x_4 + x_5 = 5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

consider the set of columns $[\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5]$. Does it determine a basis? If so is it a feasible basis? Is it optimal?

Exercise 6*

In class, we proved that the (mininum) vertex covering problem and the (maximum) matching problem are a weak dual pair. Prove that for bipartite graphs they, actually, are a strong dual pair.

Exercise 7*

Generalized Assignment Problem. Suppose there are n types of tracks available to delivery products to m clients. The cost of track of type i serving client j is c_{ij}. The capacity of track type i is C_i and the demand of each client is d_j. There are a_i tracks for each type. Formulate an IP model to decide how many tracks of each type are needed to satisfy all clients so that the total cost of doing the deliveries is minimized. If all the input data will be integer, will the solution to the linear programming relaxation be integer?

Exercise 8*

Consider the following three matrices:

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

For each of them say if it is totally unimodular and justify your answer.

Exercise 9*

In class we stated that for the uncapaciteted facility location problem there are two formulations:

$$X = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{B}^1 : \sum_{i=1}^{m} x_i \leq my, x_i \leq 1 \text{ for } i = 1, \dots, m\}$$

$$P = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^1 : x_i \leq y \text{ for } i = 1, \dots, m, y \leq 1\}$$

Proove that the polyhedron P describes conv(X). [Hint: use the TUM theory.]

Exercise 10

1. Prove that the polyhedron $P = \{(x_1, \dots, x_m, y) \in \mathbb{R}^{m+1} : y \leq 1, x_i \leq y \text{ for } i = 1, \dots, m\}$ has integer vertices. [Hint: start by writing the constraint matrix.]

2. Consider the following (integer) linear programming problem:

$$\begin{aligned} \min\ & c_1x_1 + c_2x_2 + c_3x_3 + c_3x_4 \\ & x_3 + x_4 \geq 10 \\ & x_2 + x_3 + x_4 \geq 20 \\ & x_1 + x_2 + x_3 + x_4 \geq 30 \\ & x_2 + x_3 \geq 15 \\ & x_1, x_2, x_3, x_4 \in \mathbb{Z}^+_0 \end{aligned} \tag{1}$$

   The constraint matrix has consecutive 1's in each column. Matrices with consecutive 1's property for each column are totally unimodular. Show that this fact holds for the specific numerical example

(1). That is, show first that the constraint matrix of the problem has consecutive 1s in the columns and then that you can transform this matrix into one that you should recognize to be a TUM matrix. [Hint: rewrite the problem in standard form (that is, in equation form) and add a redundant row $\mathbf{0} \cdot \mathbf{x} = 0$ to the set of constraints. Then perform elementary row operations to bring the matrix to a known form.]

3. Use one of the two previous results to show that the *shift scheduling problem* in Exercise 1 of this Sheet can be solved efficiently when formulated as a mathematical programming problem. (You do not need to find numerical results.)

Exercise 11*

This is a continutation of the Factory Planning problem from the computer lab class Sheet 5. The setting is the multiperiod problem discussed in tasks 2 and 3.
Here, instead of stipulating when each machine is down for maintenance, it is desired to find the best month for each machine to be down.
Each machine must be down for maintenance in one month of the six apart from the grinding machines, only two of which need be down in any six months.
Extend the model that correctly addressed tasks 2 and 3 to allow it to make these extra decisions.

- How many variables did you need to add? What is the domain of these variables?
- Is the solution from Task 3 a valid solution to this problem? What information can it bear in this new case?
- Implement and solve the model in Python and Gurobi. After how many nodes in the branch and bound tree is the optimal solution found? And after how many is it proven optimal?
- How much worth is the extra flexibility of choosing when to place downtimes?