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DM559/DM545 – Linear and integer programming

Sheet 10, Spring 2018 [pdf format]

Solution:
Contains Solutions!

Exercise 1*

Given the Network in Figure 1, determine the max flow and indicate the min cut.

Exercise 2*

Solve the following IP problem with Gomory's fractional cutting plane algorithm, indicating the cut inequalities in the space of the original variables

$$\begin{aligned} \max \;\; & x_1 + 2x_2 \\ & x_1 - 2x_2 \geq -2 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

Solution:
We represent the situation graphically using the LP Grapher tool linked from the web page (any other graphing tool like Grapher under Macosx would do a similar job)

```
|-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    -z |     b |
|-------+-------+-------+-------+-------+-------+
|    -1 |     2 |     1 |     0 |     0 |     2 |
|     1 |     1 |     0 |     1 |     0 |     3 |
|-------+-------+-------+-------+-------+-------+
|     1 |     2 |     0 |     0 |     1 |     0 |
|-------+-------+-------+-------+-------+-------+
pivot column: 2
```

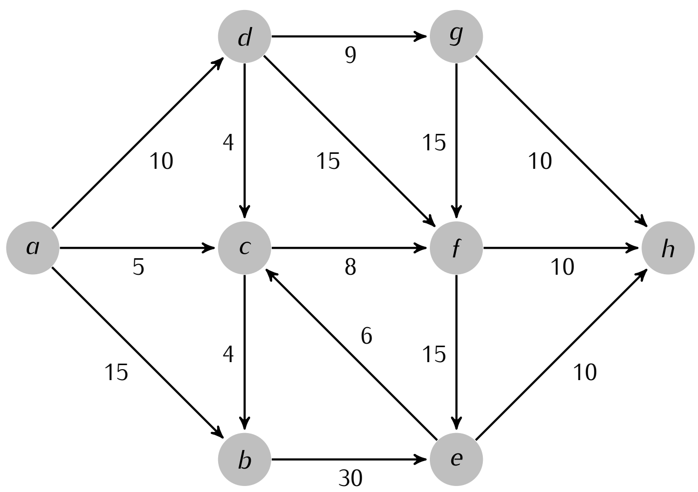
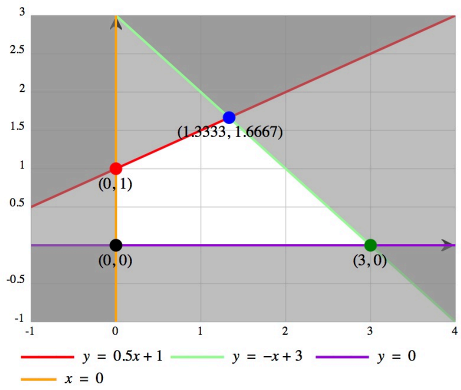


Figure 1: Find the maximum flow from a to h. Numbers on arcs are capacity values.




```
pivot row: 1
 pivot: 2

|-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    -z |     b |
|-------+-------+-------+-------+-------+-------+
|  -1/2 |     1 |   1/2 |     0 |     0 |     1 |
|   3/2 |     0 |  -1/2 |     1 |     0 |     2 |
|-------+-------+-------+-------+-------+-------+
|     2 |     0 |    -1 |     0 |     1 |    -2 |
|-------+-------+-------+-------+-------+-------+
pivot column: 1
pivot row: 2
 pivot: 3/2

|-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    -z |     b |
|-------+-------+-------+-------+-------+-------+
|     0 |     1 |   1/3 |   1/3 |     0 |   5/3 |
|     1 |     0 |  -1/3 |   2/3 |     0 |   4/3 |
|-------+-------+-------+-------+-------+-------+
|     0 |     0 |  -1/3 |  -4/3 |     1 | -14/3 |
|-------+-------+-------+-------+-------+-------+
```

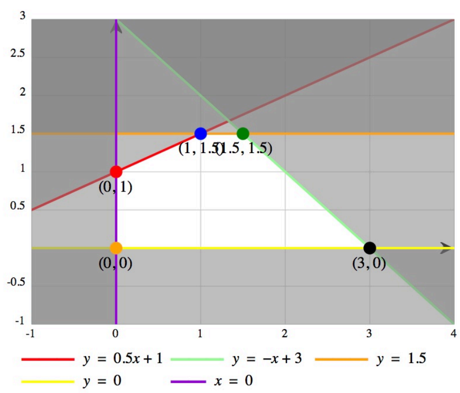
We choose the first row. The cut is:

$$1/3x_3 + 1/3x_4 \geq 2/3$$

In the original variables it is:

$$x_2 \leq 1$$

The cut separates the LP solution as evident from the picture below:



We insert the cut in the simplex and proceed by dual simplex:

```
|-------+-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    x5 |    -z |     b |
|-------+-------+-------+-------+-------+-------+-------+
|     0 |     0 |  -1/3 |  -1/3 |     1 |     0 |  -2/3 |
|     0 |     1 |   1/3 |   1/3 |     0 |     0 |   5/3 |
|     1 |     0 |  -1/3 |   2/3 |     0 |     0 |   4/3 |
|-------+-------+-------+-------+-------+-------+-------+
|     0 |     0 |  -1/3 |  -4/3 |     0 |     1 | -14/3 |
|-------+-------+-------+-------+-------+-------+-------+
```

Note that we could have inserted the cut also in the form $x_2 \leq 1$ but then we would have to put the tableau in canonical form.

Exercise 3*

This exercise is taken from the exam of 2012.
The Danish Research Council has to decide which research projects to finance. The total budget for the projects is 20 million Dkk. The table below shows the evaluation from 0 (worst) to 2 (best) that the projects received by the external reviewers and the amount of money required.

	1	2	3	4	5
Evaluation score	1	1.8	1.4	0.6	1.4
Investment (in million of DKK)	6	12	10	4	8

Projects 2 and 3 have the same coordinator and the Council decided to grant only one of the two.
The Council wants to select the combination of projects that will maximize the total relevance of the projects, that is, the sum of the evaluation score while remaining within the budget.

Formulate the problem of deciding on which project the Council has to invest as an integer linear programming problem P.

Solution:

In .lp format:

```
\* Problem: lp3 *\

Maximize
 tot: + x(1) + 1.8 x(2) + 1.4 x(3) + 0.6 x(4) + 1.4 x(5)

Subject To
 budget: + 6 x(1) + 12 x(2) + 10 x(3) + 4 x(4) + 8 x(5) <= 20
 a: + x(2) + x(3) <= 1

Bounds
 0 <= x(1) <= 1
 0 <= x(2) <= 1
 0 <= x(3) <= 1
 0 <= x(4) <= 1
 0 <= x(5) <= 1

End
```

We want the IP instance solved using the branch-and-bound algorithm. What is the optimal solution x^* to the LP relaxation P'? (Hint: use Gurobi Python to find out.]

Solution:

The following gurobypi model gives solution $\mathbf{x} = [1, 0.5, 0, 0, 1]$ and $z = 3.3$.

```
from gurobipy import *

m = Model("knapsack")

s=[1,1.8,1.4,0.6,1.4]
c=[6,12,10,4,8]
#x = [m.addVar(name="x") for i in range(5)]
x = {i:m.addVar(lb=0.0, ub=1.0, vtype=GRB.CONTINUOUS, name="x%d" % i)
     for i in range(1,6)}

m.update()

m.setObjective(quicksum(s[i-1]*x[i] for i in range(1,6)), GRB.MAXIMIZE)

m.addConstr(quicksum(c[i-1]*x[i] for i in range(1,6))<=20, "c1")
m.addConstr(x[2]+x[3] <= 1, "c2")

m.optimize()
print map(lambda v: v.x, m.getVars())
```

```
Optimize a model with 2 rows, 5 columns and 7 nonzeros
Coefficient statistics:
  Matrix range     [1e+00, 1e+01]
  Objective range [6e-01, 2e+00]
  Bounds range     [1e+00, 1e+00]
  RHS range        [1e+00, 2e+01]
Presolve time: 0.00s
Presolved: 2 rows, 5 columns, 7 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    6.2000000e+00   2.250000e+00   0.000000e+00      0s
       2    3.3000000e+00   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds
Optimal objective  3.300000000e+00
[1.0, 0.166, 0.0, 1.0, 1.0]
```

The rounding heuristic applied to the solution x^* gives a feasible solution x'. Which one? With the knowledge collected until this stage which of the three following statements is correct:

1. x' is certainly optimal
2. x' is certainly not optimal
3. x' might be optimal

(Remember to justify your answer.)

Solution:

The rounding heuristic updates x^* setting $x_2 = 0$ or $x_2 = 1$. The latter gives an infeasible solution while the former gives $[1, 0, 0, 1, 1]$ with value 3. We cannot say at this stage if x' is optimal because the optimality gap 3.3–3 is not closed. Hence (iii) is correct.

The two subproblems generated by the branch-and-bound algorithm after finding x^* correspond to choosing or not choosing a particular project. Which one?

Solution:
The solution is $[1, 0.166, 0, 1, 1]$ and the only fractional variable is x_2 hence we branch on it.

Suppose the branch-and-bound algorithm considers first the subproblem corresponding to not choosing this project. Let's call this subproblem and its corresponding node in the search tree SP1. What is the optimal solution to its LP relaxation?

Solution:
Adding the constraint `x2<=0` to the GLPK code above we obtain:

$$x = [1, 0, 0.2, 1, 1]$$

and $z = 3.28$.

Next, the branch-and-bound algorithm considers the subproblem corresponding to choosing the project, i.e., subproblem SP2. Find the optimal solution to its LP relaxation. Which are the active nodes (i.e., open subproblems) at this point?

Solution:
Adding the constraint `x2>=1` to the Python code above we obtain:

$$x = [0, 1, 0, 0, 1]$$

and $z = 3.2$. This is an integer solution and hence a lower bound.
Node SP2 is not active since an integer solution prunes the subtree. The other node SP1 has however still potential to find a better solution since its upper bound is $3.28 > 3.2$, hence the list of active nodes contains SP1.

How does the branch and bound end?

Solution:
We need to examine the active nodes. Hence we branch once more with $x_3 \leq 0$ (subproblem SP3) and $x_3 \geq 1$ (subproblem SP4). The LP relaxation of SP3 gives an integer solution $[1, 0, 0, 1, 1]$ of value 3 and SP4 gives $[0.33, 0, 1, 0, 1]$ of value 3.13. Hence the upper bound from subtree SP1 is 3.13 which is smaller than the lower bound 3.2 of SP2 and we can prune SP4 by bounding. The optimal solution is the one on node SP2.