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DM559/DM545 – Linear and integer programming

Sheet 12, Spring 2017 [pdf format]

The following exercises are updated versions of tasks appeared in exams 2010 and 2009.

Solution:

Contains Solutions!

Exercise 1

Consider the following IP problem:

$$\begin{aligned} \max \quad & 4x_1 + 7x_2 \\ s.t. \quad & x_1 + 3x_2 \leq 12 \\ & 4x_1 + 6x_2 \leq 27 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1, x_2 \geq 0, x_1, x_2 \in \mathbb{Z} \end{aligned} \tag{1}$$

Subtask a

Give a heuristic primal bound and describe how you determined it.

Solution:

$x = [0, 0]$ is feasible because it satisfies all constraints and has value $z = 0$. This is a lower bound to the optimal solution.

Subtask b

Write the LP relaxation (2lp) of (2) to obtain a dual bound. Explain the relation between the optimal solution of (2lp) and the optimal solution of (2).

Solution:

We relax x_1 and x_2. The problem (2lp) becomes:

$$\begin{aligned} \max \quad & z_{LP} = 4x_1 + 7x_2 \\ s.t. \quad & x_1 + 3x_2 \leq 12 \\ & 4x_1 + 6x_2 \leq 27 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{2}$$

(2lp) gives an upper bound to the problem (2).

Subtask c

Write the first simplex tableau of (2lp) and indicate which variables constitute a basic solution. Call s_1, s_2, s_3 the slack variables.

Subtask d

Explain which variable leaves the basis and which variable enters the basis in the first iteration of the simplex algorithm with largest coefficient pivot rule. Show that the answer would be the same if, instead, the largest increase pivot rule was used.

Subtask e

After a number of iterations the tableau is the following:

```
x1      x2      s1      s2      s3      -z      b
0       1       2/3     -1/6    0       0       7/2
1       0       -1      1/2     0       0       3/2
0       0       8/3     -5/3    1       0       7
0       0       -2/3    -5/6    0       1       -61/2
```

Argue that an optimal solution for (2lp) has been found and give for it the value of x_1 and x_2 together with its objective function value. Report the optimality gap for (2) at this stage.

Subtask e

Show how you can reconstruct the tableau at the previous point by just knowing that x_2, x_1 and s_3 are in basis and that:

$$A_B^{-1} = \begin{bmatrix} 2/3 & -1/6 & 0 \\ -1 & 1/2 & 0 \\ 8/3 & -5/3 & 1 \end{bmatrix}.$$

Solution:

```
import numpy as np
from fractions import Fraction as f
A=np.array([[ 1, 3,0],[4, 6,0],[4,2,1]])
# print np.linalg.inv(A)
A_1= np.array([[f(2,3),f(-1,6),0],[f(-1),f(1,2),0],[f(8,3),f(-5,3),1]])
print np.dot(A[:,[1,0,2]],A_1)
```

Subtask f

From the second row of the last tableau derive a Gomory cut and write it in the space of the original variables.
Argue shortly that the cut is a valid inequality for (2) and that it will make the current optimal solution of (2lp) infeasible.

Subtask g

Introduce the cut in the tableau and explain how the solution algorithm will continue. Indicate the new pivot and explain how you found it. (You do not need to carry out the simplex iteration.)

Subtask h

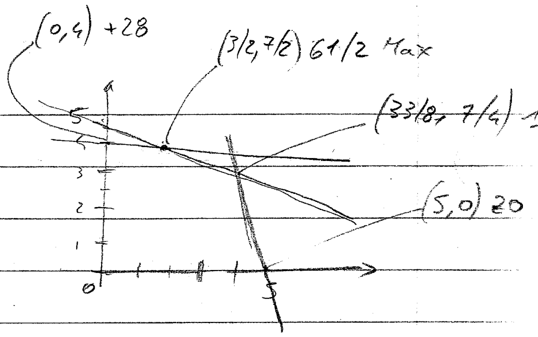
After the introduction of the cut the tableau of the optimal solution to the new LP problem is the following.

```
x1      x2      s1      s2      s3      s4      -z      b
0       1       2/3     0       0       -1/3    0       11/3
0       0       0       1       0       -2      0       1
0       0       8/3     0       1       -10/3   0       26/3
1       0       -1      0       0       1       0       1
0       0       -2/3    0       0       -5/3    1       -89/3
```

Explain how the solution process would continue from this stage by branch and bound. Define the next branching and indicate what can be done in each open node.

Solution:

TASK 1



$\max z = 4x_1 + 7x_2$

$x_1 + 3x_2 \leq 12$

$-4x_1 + 6x_2 \leq 27$

$4x_1 + 2x_2 \leq 20$

(a) $(0,0)$ is feasible $\Rightarrow z = 0 = LB$

(c)

(d)

x_1	x_2	s_1	s_2	s_3	$-z$	b	b_i/a_{i1}	b_i/a_{i2}	$c_j \cdot b_i/a_{ij}$
1	3	1	0	0	0	12	12	4	$7 \cdot 4 = 28$
4	6	0	1	0	0	27	$27/4 \simeq 8$	$\frac{27}{6} \simeq 4$	
4	2	0	0	1	0	20	5	10	$5 \cdot 4 = 20$
4	7	0	0	0	1	0			

largest coefficient $\leadsto 7$

largest increase $\leadsto 7$

1/3	1	1/3	0	0	0	4
2	0	-2	1	0	0	3
10/3	0	-2/3	0	1	0	12
-5/3	0	7/3	0	0	1	28

(e)

0	1	2/3	-1/6	0	0	7/2
1	0	-1	1/2	0	0	3/2
0	0	8/3	-5/3	1	0	7
0	0	-2/3	-5/6	0	1	-61/2

$GAP = \frac{61/2 - 0}{61/2} = 1$

(f) I row: $\frac{2}{3}s_1 + \frac{5}{6}s_2 \geq \frac{1}{2}$ $\qquad \frac{1}{2}s_1 \geq \frac{1}{2}$

$$\frac{2}{3}(12 - x_1 - 3x_2) + \frac{5}{6}(27 - 4x_1 - 6x_2) \geq \frac{1}{2}$$

$$8 - \frac{2}{3}x_1 - 2x_2 + \frac{135}{6} - \frac{10}{3}x_1 - 5x_2 \geq \frac{1}{2}$$

$$-4x_1 - 7x_2 \geq \frac{1}{2} - 8 - \frac{135}{6}$$

II row: $\frac{1}{2}s_2 \geq \frac{1}{2}$ $\qquad$ (i)-form

$$\frac{1}{2}(27 - 4x_1 - 6x_2) \geq 1$$

$$-4x_1 - 6x_2 \geq -26$$

$$-2x_1 - 3x_2 \geq -13$$

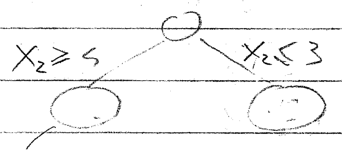
$$2x_1 + 3x_2 \leq 13$$ (ii)-form

(g) Introducing the i-form we have a basis but infeasible.
Introducing the ii-form we do elementary row operations to arrive to a canonical form with infeasible basis.
We use dual simplex:

$$\begin{array}{cccccccc}
0 & 0 & 0 & \boxed{-\frac{1}{2}} & 0 & 0 & 1 & -\frac{1}{2} \\
0 & 1 & \frac{2}{3} & -\frac{1}{6} & 0 & 0 & 0 & \frac{7}{2} \\
1 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{2} \\
0 & 0 & \frac{8}{3} & -\frac{5}{3} & 1 & 0 & 0 & 7 \\
0 & 0 & -\frac{2}{3} & -\frac{5}{6} & 0 & 1 & 0 & -\frac{61}{2}
\end{array}$$

1) pivot < 0
2) row with b term negative
3) col that $\min\left|\frac{c_j}{a_{ij}}\right|$ $\Rightarrow$ $-\frac{1}{2}$ is pivot

(h) Using branch and bound on we have



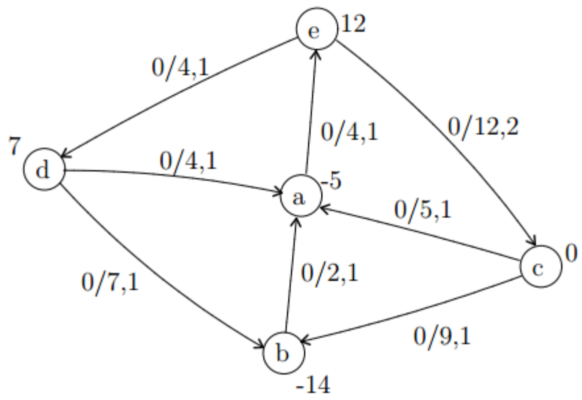
we could use heuristics again at each node

the LP requires a dual-simplex step
sol. will be $(0,4)$ of val 28

sol will be $(2,3)$ of val $\underline{\underline{29}}$

Exercise 2

Let $\mathcal{N} = (V, A, l, u, c, b)$ be the network depicted below. Numbers on vertices correspond to the balance values, numbers on each arc ij correspond to $l_{ij}/u_{ij}, c_{ij}$.



Subtask a

Let $\mathbf{x}$ be the flow made by $x_{ed} = 3, x_{da} = 4, x_{db} = 6, x_{ae} = 0, x_{ba} = 1, x_{ca} = 0, x_{cb} = 9, x_{ec} = 9$. Show that it is feasible and evaluate its cost.

Subtask b

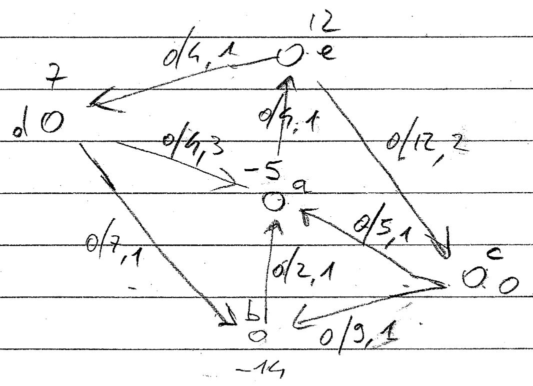
Find a feasible flow x^* in $\mathcal{N}$ of minimum cost with linear programming. Draw x^* in $\mathcal{N}$ and give its value.

Solution:

The solution was obtained using max flow algorithms that in 2017 we did not treat. Hence the only relevant aspect in the solution is the final numerical result. Here it was expected to find that result using an LP model and solving it with gurobi.

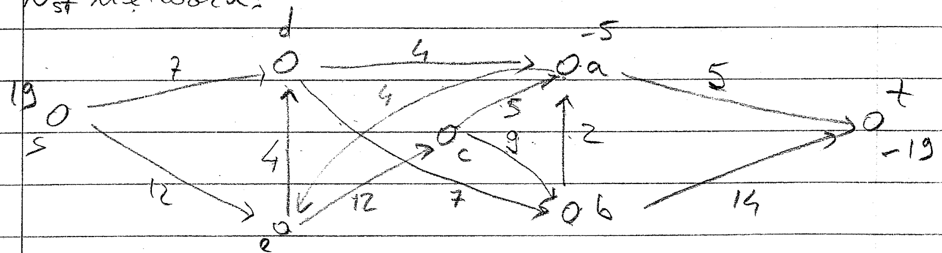
TASK 2

(a)

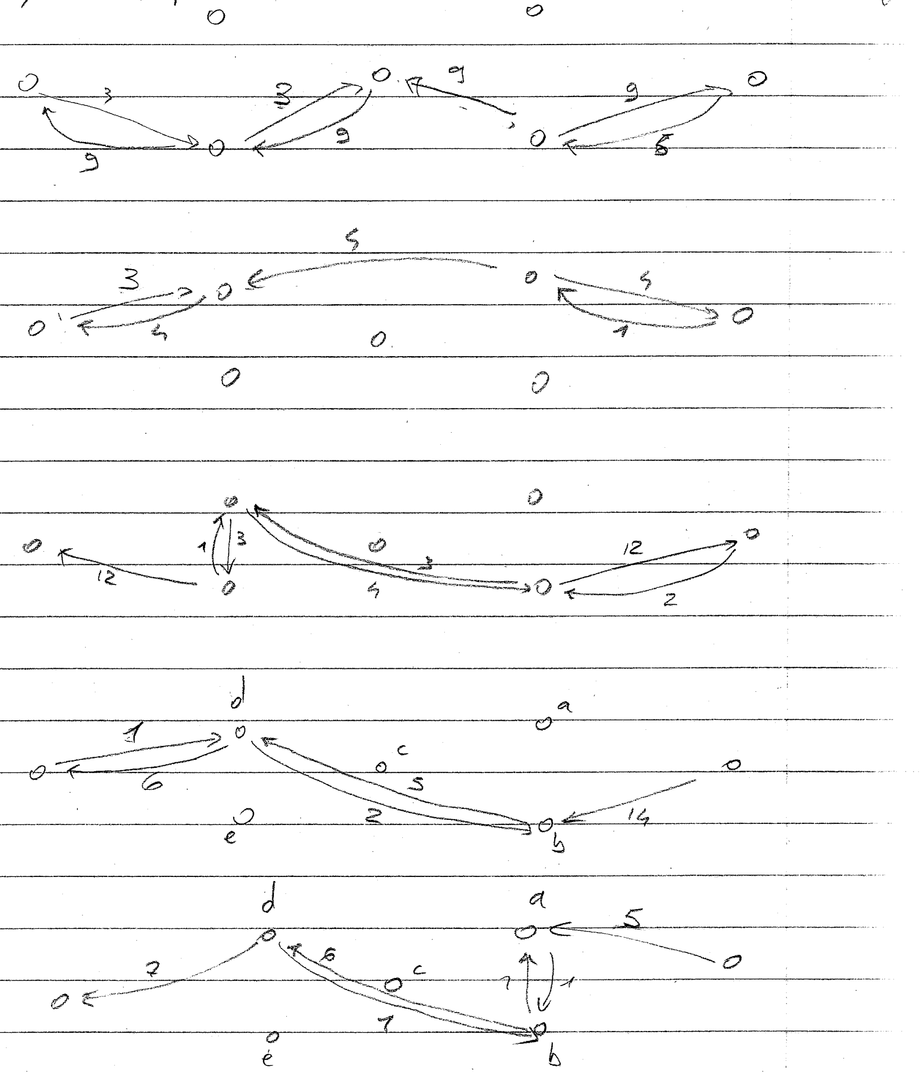


$l=0$

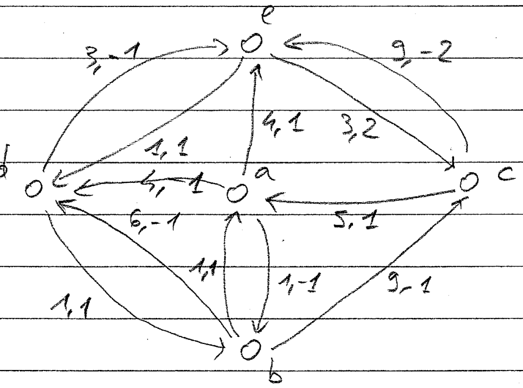
N_{st} network:



Apply max flow to N_{st} network. Ford Fulkerson alg:

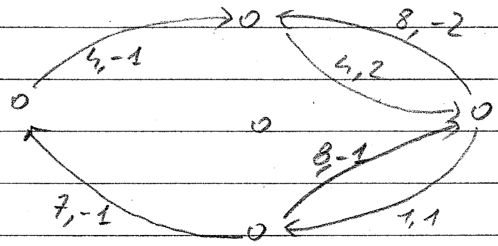


(b) $N(x)$

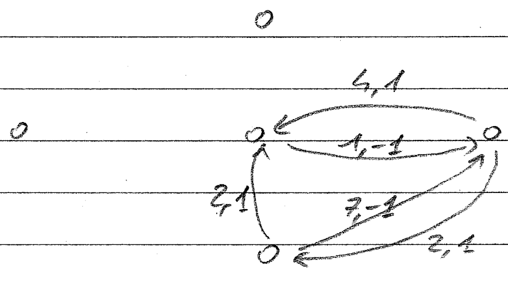


cost 41

(c)

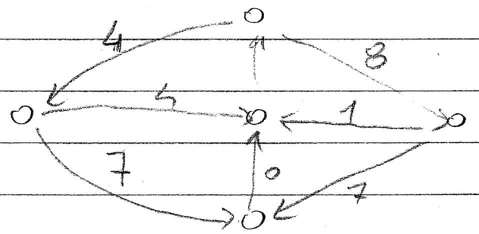


cost 40



cost 39

final flow



Exercise 3

The matrix description of the simplex method is at the basis of the *revised simplex method* that is implemented in computer programs for solving LP problems. In the revised simplex method the iterations are not carried out on the tableau but, instead, faster updates are performed by keeping the basis matrix and its inverse.

For the LP problem

$$\begin{aligned} \max \quad & x_1 + 2x_2 + x_3 + x_4 \\ & 2x_1 + x_2 + 5x_3 + x_4 \leq 8 \\ & 2x_1 + 2x_2 + 4x_4 \leq 12 \\ & 3x_1 + x_2 + 2x_3 \leq 18 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned} \tag{3}$$

the inverse of the basis matrix for the basic variables $\mathbf{x}_B = (x_3, x_2, s_3)$ and non-basic variables $\mathbf{x}_N = (x_1, x_4, s_1, s_2)$, where s_1, s_2, s_3 are slack variables, is

$$A_B^{-1} = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.0 & 0.5 & 0 \\ -0.4 & -0.3 & 1 \end{bmatrix}$$

Subtask a

Using only matrix operations, like in the revised simplex method, show that the given basis structure corresponds to an optimal solution and give the value of its variables.

Subtask b

What is the value of the simplex multipliers and of the optimal solution of the dual problem?

Subtask c

Are all constraints "binding", or is there slack capacity in some of them? Answer this question in the light of the complementary slackness theorem.

Subtask d

Explain how you determine whether the basis (x_3, x_2, s_3) remains optimal if the second constraint's right-hand side is changed to 26. In case it remains optimal, determine the change in the optimal objective value.

Subtask e

What will be the new basis structure if the objective function is changed to

$$3x_1 + 2x_2 + x_3 + x_4?$$

Explain how you determine the entering and the leaving variables.

Subtask f

A new variable is introduced with coefficients of 3, 5, 6 in the first, second and third constraint, respectively. Determine what cost coefficient should the new variable have in order to have a change in the structure of the optimal basis.

Solution:

TASK 3

(a)

$$\begin{array}{cc|c} A_B & A_N & b \\ c_B & c_N & \end{array}$$

$$A_B x_B + A_N x_N = b$$

$$x_B = A_B^{-1} b$$

$$\bar{c}_B^T = c_B^T - \pi A_B = 0 \Rightarrow c_B^T A_B^{-1} = \pi$$

$$\bar{c}_N^T = c_N^T - \pi A_N = c_N^T - c_B^T A_B^{-1} A_N$$

$$\begin{array}{ccccccccc} & x_2 & x_3 & & & & S_3 & & \\ \end{array}$$
$$\begin{bmatrix} 2 & 1 & 5 & 1 & 1 & 0 & 0 & 0 & 8 \\ 2 & 2 & 0 & 4 & 0 & 1 & 0 & 0 & 12 \\ 3 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 18 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_B = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \qquad A_B^{-1} = \begin{bmatrix} \frac{2}{10} & 0 & -\frac{4}{10} \\ -\frac{1}{10} & \frac{5}{10} & -\frac{3}{10} \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0{,}2 & -0{,}1 & 0 \\ 0 & 0{,}5 & 0 \\ -0{,}4 & -0{,}3 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} 0{,}2 & -0{,}1 & 0 \\ 0 & 0{,}5 & 0 \\ -0{,}4 & -0{,}3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1{,}6 - 1{,}2 \\ 6 \\ -3{,}2 - 3{,}6 + 18 \end{bmatrix} = \begin{bmatrix} 0{,}4 \\ 6 \\ 11{,}2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ S_1 \end{bmatrix}$$

$$\bar{c}_N^T = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0{,}2 & -0{,}1 & 0 \\ 0 & 0{,}5 & 0 \\ -0{,}4 & -0{,}3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix} =$$

$$= \text{"} - \text{"} \begin{bmatrix} \cancel{0{,}4 - 0{,}2} & \cancel{0{,}2 - 0{,}4} & \cancel{0{,}2} & \cancel{-0{,}1} \\ 1 & 2 & 0 & 0{,}5 \\ -0{,}8 - 0{,}6 + 3 & -0{,}4 - 1{,}2 & -0{,}4 & -0{,}3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0,2 & -0,2 & 0,2 & -0,1 \\ 1 & 2 & 0 & 0,5 \\ 1,6 & -1,6 & -0,4 & -0,3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0,2+2 & -0,2+4 & 0,2 & -0,1+1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1,2 & -2,8 & -0,2 & -0,9 \end{bmatrix}$$

All red. costs are negative $\Rightarrow$ sol. is optimal.

b) $y^T = c_B^T A_B^{-1} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} = \begin{bmatrix} 0,2 & 0,9 & 0 \end{bmatrix}$

The dual variables are the reduced costs of slack variables with change of sign. (This has been shown in the proof of the Strong duality Theorem)

$$y^T = \begin{bmatrix} 0,2 & 0,9 & 0 \end{bmatrix}$$

c) Looking at the dual var, the first and second constr. are binding. The third has a slack:

$$b_i - \sum_{j=1}^{m} a_{ij} x_j > 0$$

d) $z^T = y^T b = \begin{bmatrix} 0,2 & 0,9 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 26 \\ 18 \end{bmatrix} = 1,6 + \quad + 0 = 23,4$

new obj

before it was

$$c^T x^* = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 0,4 \\ 0 \\ 0 \\ 11,2 \end{bmatrix} = 12 + 0,4 = 12,4$$

hence obj increases

But we must make sure the b term does not become negative.

$$\bar{b} = A_B^{-1} b = \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 26 \\ 18 \end{bmatrix} = \begin{bmatrix} 1,6 - 2,6 \\ 13 \\ -3,2 - 7,8 + 18 \end{bmatrix} = \begin{bmatrix} -1 \\ 13 \\ 7 \end{bmatrix}$$

one term becomes negative ⇒ we need to iterate.

(e) we look back at how we computed red costs.

$$\bar{c}_N = [3 \quad 1 \quad 0 \quad 0] - [2,2 \quad 3,8 \quad 0,2 \quad 0,9] =$$
$$= [0,8 \; \ldots \;]$$

the red cost of x_1 becomes positive ⇒ x_1 enters the basis.
To determine the leaving var we need to see how much we can increase:

$$x_B = x_B^* - A_B^{-1} A_N x_N = x_B^* - A_B^{-1} a_e x_N$$

entering col.

$$A_B^{-1} \cdot a = \begin{bmatrix} 0,2 & -0,1 & 0 \\ 0 & 0,5 & 0 \\ -0,4 & -0,3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0,4 - 0,2 \\ 1 \\ -0,8 - 0,6 + 3 \end{bmatrix} = \begin{bmatrix} 0,2 \\ 1 \\ 1,6 \end{bmatrix}$$

$$x_B = \begin{bmatrix} 0,4 \\ 6 \\ 11,2 \end{bmatrix} - \begin{bmatrix} 0,2 \\ 1 \\ 1,6 \end{bmatrix} t \geq 0$$

$t \leq 0,4/0,2 = 0,2 \leftarrow$
$t \leq 6$
$t \leq 11,2/1,6 = 7$

x_3 leaves the basis

(f) We need to determine the red. cost for the new var.

$$\bar{c}_N^T = c_N^T - \pi A_N =$$

$$= \begin{bmatrix} \delta & 1 & 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0{,}2 & 0{,}9 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 & 1 & 0 \\ 5 & 2 & 4 & 0 & 1 \\ 6 & 3 & 0 & 0 & 0 \end{bmatrix} =$$

$$= \delta - 0{,}6 - 4{,}5 > 0$$

$$\delta > 5{,}1$$

Exercise 4

Given the following LP problem:

$$\begin{aligned} \max\ & z = x_1 - 2x_2 \\ \text{s.t.}\ & x_1 + 2x_2 - x_3 \geq 0 \\ & 4x_1 + 3x_2 + 4x_3 \leq 3 \\ & 2x_1 - x_2 + 2x_3 = 1 \\ & x_2, x_3 \geq 0 \end{aligned}$$

Subtask a

Derive the dual problem using the Lagrange multipliers approach and report the steps of the derivation.

Subtask b

Without using the simplex algorithm, determine the optimal solution of the dual knowing that the optimal solution of the primal is $x^* = (1/2, 0, 0)$.

Solution:

TASK 5

(a)

$$(P)\quad \begin{aligned} \max\ & x_1 - 2x_2 \\ & x_1 + 2x_2 - x_3 \geq 0 \\ & 4x_1 + 3x_2 + 4x_3 \leq 3 \\ & 2x_1 - x_2 + 2x_3 = 1 \\ & x_2, x_3 \geq 0 \\ & x_1 \in \mathbb{R} \end{aligned}$$

We want an UB; we bring (relax) constraints in obj. func.

$$P(y_1,y_2,y_3) = \max\ x_1 - 2x_2 + y_1(x_1 + 2x_2 - x_3) + y_2(4x_1 + 3x_2 + 4x_3 - 3) + y_3(2x_1 - x_2 + 2x_3 - 1)$$

$$\forall y_1 \geq 0,\ y_2 \lesseqgtr 0,\ y_3 \in \mathbb{R} \qquad P(y_1,y_2,y_3) \geq opt(P)$$

(this proves the weak duality thr. by construction)

$$\min_{y_1 y_2 y_3} P(y_1\, y_2\, y_3)$$ this will give us the best UB.

$$\max\ (1 + y_1 + 4y_2 + 2y_3)x_1 + (-2 + 2y_1 + 3y_2 - y_3)x_2 + (-y_1 + 4y_2 + 2y_3)x_3 - 3y_2 - y_3$$

This problem can be solved by inspection.
It is of the form:

$\max\ c_1x_1 + c_2x_2 + \dots + c_nx_n \;\Big|\Rightarrow$
- if $c_i > 0 \wedge x_i \geq 0 \Rightarrow x_i = +\infty$
- if $c_i \leq 0 \wedge x_i \geq 0 \Rightarrow x_i = 0$, bou
- if $c_i \neq 0 \wedge x_i \in \mathbb{R} \Rightarrow$ unbounded
- if $c_i = 0 \wedge x_i \in \mathbb{R} \Rightarrow$ bounded

We are only interested in the cases where the problem is bounded hence:

$$\begin{aligned} \min\ & -3y_2 - y_3 \\ & y_1 + 4y_2 + 2y_3 = -1 \\ & 2y_1 + 3y_2 - y_3 \leq 2 \\ & -y_1 + 4y_2 + 2y_3 \leq 0 \\ & y_1 \geq 0 \\ & y_2 \leq 0 \\ & y_3 \in \mathbb{R} \end{aligned} \quad \equiv \quad \begin{aligned} \min\ & +3y_2 + y_3 \\ & -y_1 + 4y_2 - 2y_3 = -1 \\ & -2y_1 - 3y_2 + y_3 \leq 2 \\ & +y_1 - 4y_2 + 2y_3 \leq 0 \\ & y_1 \lesseqgtr 0 \\ & y_2 \geq 0 \\ & y_3 \in \mathbb{R} \end{aligned}$$

Exercise 5 Formulering IP problemer og cutting plane metoden

En digraf $D = (V, A)$ er stærkt sammenhængende, hvis der for alle par af punkter $x, y \in V$ gælder, at D indeholder en ensrettet vej fra x til y og en ensrettet vej fra y til x. Dette er ækvivalent med at enhver ikke tom ægte delmængde X af V har mindst en kant ud af sig (dvs en kant ij hvor $i \in X$ og $j \notin X$). MSSS[1] problemet er som følger: Givet en stærkt sammenhængende digraf $D = (V, A)$; find en udspændende[2] delgraf $D' = (V, A')$ af D, så D' også er stærkt sammenhængende og har så få kanter som muligt.

Spørgsmål a:

Formuler MSSS problemet som et heltalsprogrammeringsproblem. Du skal redegøre for at din formulering er korrekt.

Spørgsmål b:

Beskriv kort hvordan man kan bruge flows til at finde en mindste mændge af kanter $A'' \subseteq A$ fra en stærkt sammenhængende digraf $D = (V, A)$ så $d^-_{A''}(v) \geq 1$ og $d^+_{A''}(v) \geq 1$ for alle $v \in V$. Her er $d^-_{A''}(v)$ ($d^+_{A''}(v)$) antallet af kanter fra A'' som går ind i (ud fra) v.

Spørgsmål c:

Gør kort rede for, hvordan man kunne løse LP-relaksationen af MSSS problemet ved hjælp af en cutting plane metode. Du skal gøre rede for, hvordan man, ved hjælp af flows, kan checke om den aktuelle LP-løsning overholder alle de oprindelige uligheder, samt beskrive hvad man gør hvis den ikke overholder dem alle sammen. Hint: Du kan foreksempel tilføje de redundante uligheder der udtrykker, at ethvert punkt $i \in V$ skal have mindst en kant ud fra sig og mindst en kant ind til sig i D' og starte med kun disse uligheder.

Spørgsmål d:

Giv et eksempel som viser, at den optimale løsning til LP-relaksationen af MSSS problemet ikke behøver være heltallig og forklar kort hvordan man kan komme videre (mod en optimal heltalsløsning) i dette tilfælde.

[1] Minimum Spanning Strong Subdigraph
[2] dvs den indeholder alle punkterne fra D

Exercise 6 Branch and Bound

This exercise provides an example of Branch and Bound algorithm independent on linear programming. Consider the instance of TSP with 5 nodes and 10 edges that is showin in Figure 1. The tour $T = 1, 3, 5, 2, 4, 1$ has cost 17 and is used as initial solution.

Spørgsmål a:

The 1-tree bound for the TSP is based on the following observation. If we select some node of the instance, say node 1, then a tour consists of a special spanning tree (namely a path) on the remaining $N - 1$ nodes plus two edges connecting node 1 to this spanning tree. Hence we obtain a relaxation of the TSP if we take as feasible solutions arbitrary spanning trees on the node set $\{2, \ldots, N\}$ plus two additional edges incident to node 1. Such a graph contains precisely one circle and is called 1-tree. To get the optimum 1-tree we calculate the minimum spanning tree (MST) on the node set $\{2, \ldots, N\}$, which can be done in polynomial time using Prim's algorithm. Node 1 is then connected to its nearest and second nearest neighbor on the MST.
Find the optimalt 1-tree for the given instance when the special node is node 1.

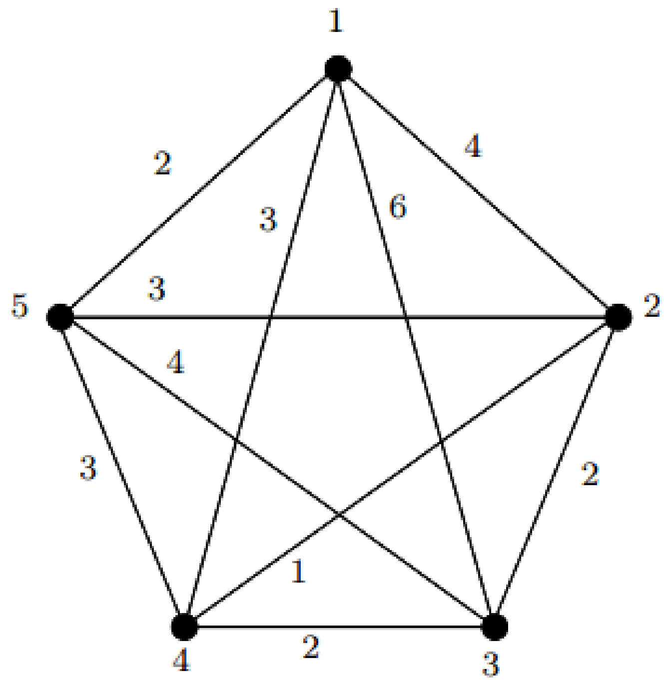


Figure 1: En TSP instans

Spørgsmål b:

Løs TSP problemet fra Figur 1 til optimalitet ved hjælp af branch and bound, hvor du bruger 1-træer som nedre grænse (det er altid knude 1 der er den specielle knude) og starter med turen T som en kendt øvre grænse. Forgreningen i branch and bound træet skal foretages ved at udelukke kanter ved et udvalgt punkt som er endepunkt for mindst 3 kanter i 1-træet. **Ved det første 1-træ skal du forgrene ud fra kanterne som er incidente med punkt 4.** Du skal også beregne den nedre grænse for alle de nye knuder i branch and bound træet, så snart du laver dem og bruge disse til at begrænse antallet af delproblemer du fortsætter med. Husk, at du skal vælge lovlige 1-træer i hvert skridt. Hvis der er flere muligheder må du gerne vælge et sådant som giver dig bedst mulig information.

Forklar kort (med begrundelser) hvad du konkluderer i de enkelte skridt, hvilke knuder i branch and bound træet du er nød til at forsætte fra (branche) og hvilke du kan afslutte.