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DM559/DM545 – Linear and integer programming

Sheet 5, Spring 2017 [pdf format]

Solution:
Included.

Exercise 1*

Show that the dual of $\max\{c^Tx \mid Ax = b, x \geq 0\}$ is $\min\{y^Tb \mid y^TA \geq c\}$.

Solution:
This was shown in class by means of the Lagrangian approach.
Let's show it here by the bounding method.
Given $\max\{c^Tx \mid Ax = b, x \geq 0\}$ we search for multipliers $y \in \mathbb{R}^n$ such that $y^TAx = y^Tb$ (since we have equalities, the multipliers can be both positive or negative as we do not need to ensure the maintainance of the direction of the inequality). To ensure that we find an upper bound and hence have $c^Tx \leq y^TAx$, we impose $y^TA \geq c^T$ (since $x \geq 0$). Hence, the best upper bound will be given by solving $\min\{y^Tb \mid y^TA \geq c^T\}$ (recalling from linear algebra that $(AB)^T = B^TA^T$, we can rewrite: $\min\{y^Tb \mid A^Ty \geq c\}$, which is the form we would obtain using the recipe method.)

Exercise 2*

Consider the following LP problem:

$$
\begin{aligned}
\max\ & 2x_1 + 3x_2 \\
& 2x_1 + 3x_2 \leq 30 \\
& x_1 + 2x_2 \geq 10 \\
& x_1 - x_2 \leq 1 \\
& x_2 - x_1 \leq 1 \\
& x_1 \geq 0
\end{aligned}
$$

- Write the dual problem
- Using the optimality conditions derived from the theory of duality, and without using the simplex method, find the optimal solution of the dual knowing that the optimal solution of the primal is $(27/5, 32/5)$.

Solution:
The dual is:

$$
\begin{aligned}
\max\ & 30y_1 + 10y_2 + y_3 + y_4 \\
& 2y_1y_2 + y_3 - y_4 \geq 2 \\
& 3y_1 + 2y_2 - y_3 + y_4 = 3 \\
& y_1, y_3, y_4 \geq 0 \\
& y_2 \leq 0
\end{aligned}
$$

We use the complementary slackness theorem.

$$
\begin{cases}
2y_1y_2 + y_3 - y_4 = 2 \\
3y_1 + 2y_2 - y_3 + y_4 = 3 \\
y_2 = 0 \\
y_3 = 0
\end{cases}
$$

The first because the corresponding variable of the primal is ¿0, the second for the same reason or however because it is already tight by definition, the third and fourth equation are a consequence of the fact that substituting the value of the primal variables variables in the primal problem, the second and third constraints are binding. What we obtain is a linear system of four equations in four variables that we can solve to find the value of the variables of the dual problem.

Exercise 3*

Consider the problem

$$\begin{array}{ll} \text{maximize} & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Without applying the simplex method, how can you tell whether the solution $(2, 0, 1)$ is an optimal solution? Is it? [Hint: consider consequences of Complementary slackness theorem.]

Exercise 4*

Consider the following problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 2 \\ & 2x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

In the ordinary simplex method this problem does not have an initial feasible basis. Hence, the method has to be enhanced by a preliminary phase to attain a feasible basis. Traditionally we talk about a *phase I–phase II* simplex method. In phase I an initial feasible solution is sought and in phase II the ordinary simplex is started from the initial feasible solution found.
There are two ways to carry out phase I.

- In lecture 4 we saw a way to find an initial feasible basis via an auxiliary LP problem defined by introducing auxiliary variables and minimizing them in the objective. Phase I is thus carried out by solving an auxiliary LP problem whose solution gives an initial feasible basis or a proof of infeasibility.
- The strong duality theorem states that we can solve the primal problem by solving its dual. You can verify that applying the *primal simplex method* to the dual problem corresponds to the following method, called *dual simplex method* that works on the primal problem:

  1. (Feasibility condition) select the leaving variable by picking the basic variable whose right-hand side term is negative, i.e., select i^* with $b_{i^*} < 0$.
  2. (Optimality condition) pick the entering variable by scanning across the selected row and comparing ratios of the coefficients in this row to the corresponding coefficients in the objective row, looking for the largest negated. Formally, select j^* such that $j^* = \min\{|c_j/a_{i^*j}| : a_{i^*j} < 0\}$
  3. Update the tableau around the pivot in the same way as with the primal simplex.
  4. Stop if no right-hand side term is negative.

Opposite to the primal simplex method, the dual simplex method iterates through infeasible basis solutions, while maintaining them optimal, and stops when a feasible solution is reached.
Duality can help us with the issue of initial feasible basis solutions. In the problem above, if the objective function was $w = -x_1 - x_2$, then the initial basis solution of the dual problem would be feasible and we could solve the problem solving the dual problem with the primal simplex. But with objective function z the simplex has infeasible initial basis in both problems. However we can change temporarily the objective function z with w and apply the dual simplex method. When it stops we reached a feasible solution that is optimal with respect to w. We can then reintroduce the original objective function and continue iterating with the primal simplex. This phase I–phase II simplex method is also called the *dual-primal simplex method*. Apply this method to the problem above and verify that it leads to the same solution as in point 1.

Solution:

$$\begin{array}{rrrrrl} \max & x_1 & - & x_2 & = & z \\ & x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & 2x_2 & \geq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

We put in equational standard form by introducing a slack variable $s_1 \geq 0$ and a surplus variable $s_2 \geq 0$:

$$\begin{array}{rrrrrrrrl} \max & x_1 & - & x_2 & = & z & & & \\ & x_1 & + & x_2 & + & s_1 & & = & 2 \\ & 2x_1 & + & 2x_2 & & & - s_2 & = & 2 \\ & & & & & & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}$$

This form is not canonical and therefore the first tableau does not have a feasible starting solution.

Auxiliary Problem Approach

We proceed by

- Phase I solving an auxiliary/augmented problem
- Phase II continuing with ordinary simplex

Phase I We introduce an auxiliary variable $a_1 \geq 0$ in the constraint that makes the infeasibility to yield a canonical form:

$$\begin{array}{rrrrrrrrrrl} \max & x_1 & - & x_2 & = & z & & & & & \\ & x_1 & + & x_2 & + & s_1 & & & & = & 2 \\ & 2x_1 & + & 2x_2 & & & - & s_2 & + a_1 & = & 2 \\ & & & & & & & & x_1, x_2, s_1, s_2, a_1 & \geq & 0 \end{array}$$

Now we have a canonical form

```
| x1 | x2 | s1 | s2 | a1 | -z | b |
|----+----+----+----+----+----+---|
|  1 |  1 |  1 |  0 |  0 |  0 | 2 |
|  2 |  2 |  0 | -1 |  1 |  0 | 2 |
|  1 | -1 |  0 |  0 |  0 |  1 | 0 |
|----+----+----+----+----+----+---|
```

This problem will have the same solution as the original one only when $a_1 = 0$. We can then solve

- an *augmented problem* by introducing the following objective function $\max w = x_1 - x_2 - Ma_1$, where M is a large enough constant or
- an *auxiliary* problem $\min w = a_1 = -\max(-a_1)$.

Let's take the auxiliary problem, if $w^* > 0$ then we will conclude that the feasibility region of the orginal problem is empty. Otherwise, if $w^* = 0$, then this implies that $a_1 = 0$ and we found a feasible solution. Let's proceed by setting up the tableau of the auxiliary problem

```
| x1 | x2 | s1 | s2 | a1 | -z | -w | b |
|----+----+----+----+----+----+----+---|
|  1 |  1 |  1 |  0 |  0 |  0 |  0 | 2 |
|  2 |  2 |  0 | -1 |  1 |  0 |  0 | 2 |
|  1 | -1 |  0 |  0 |  0 |  1 |  0 | 0 |
|  0 |  0 |  0 |  0 | -1 |  0 |  1 | 0 |
|----+----+----+----+----+----+----+---|
```

This is not in canonical form but it is easy to bring it to canonical form: just add the second row to the last one.

x1	x2	s1	s2	a1	-z	-w	b
1	1	1	0	0	0	0	2
2	2	0	-1	1	0	0	2
1	-1	0	0	0	1	0	0
2	2	0	-1	0	0	1	2

The variables s_1, a_1 give us a feasible basis now. It is not optimal. We proceed with the pivot operations. In this case it is worth noting that in the ratio rule, we do not consider the third row since that row corresponds to the orginal objective function and not to a constraint.
We make x_1 enter the basis and consequently a_1 goes out. The pivot is 2 and the new tableau:

	x1	x2	s1	s2	a1	-z	-w	b
R1'=R1-R2'	0	0	1	1/2	-1/2	0	0	1
R2'=R2/2	1	1	0	-1/2	1/2	0	0	1
R3'=R3-R2'	0	-2	0	1/2	-1/2	1	0	-1
R4'=R4-R2	0	0	0	0	-1	0	1	0

The tableau is optimal. One non basic variable has reduced cost null, which indicates that there are infinite solutions, but this is not relevant now. The relevant thing is that $w^* = 0$ hence the minimum of the auxiliary problem is 0 and hence there is a feasible solution for $a_1 = 0$. This concludes the Phase I of the algorithm since a feasible solution for the auxiliary problem is feasible also for the original problem.

Phase II We throw away the last row and the second last column from the tableau since we do not need them anymore.

x1	x2	s1	s2	a1	-z	b
0	0	1	1/2	-1/2	0	1
1	1	0	-1/2	1/2	0	1
0	-2	0	1/2	-1/2	1	-1

The tableau is not optimal. The basic solution corresponding to this tableau is feasible but not optimal. We bring s_4 in the basis and make s_3 leave. The new tableau is:

	x1	x2	s1	s2	a1	-z	b
R1'=2*R1	0	0	2	1	-1	0	2
R2'=R2+R1	1	1	0	-1/2	1/2	0	2
R3'=R3-R1	0	-2	-1	0	0	1	-2

The tableau is now optimal. The optimal solution is $\mathbf{x} = (2, 0)$ and $z^* = 2$.

Dual-Primal Simplex Method

Phase I Let's write the dual of the problem above:

$$
\begin{array}{rrcrcl}
\max & x_1 & - & x_2 & = & z \\
 & x_1 & + & x_2 & \leq & 2 \\
 & 2x_1 & + & 2x_2 & \geq & 2 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{rrcrcr}
\min & 2y'_1 & + & 2y'_2 & = & w \\
 & y'_1 & + & 2y'_2 & \geq & 1 \\
 & y'_1 & + & 2y'_2 & \geq & -1 \\
 & & & y'_1 & \geq & 0 \\
 & & & y'_2 & \leq & 0
\end{array}
\qquad
\xrightarrow[y'_2=-y_2]{y'_1=y_1}
$$

$$
\begin{array}{rrcrcr}
\min & 2y_1 & - & 2y_2 & = & w \\
 & y_1 & - & 2y_2 & \geq & 1 \\
 & y_1 & - & 2y_2 & \geq & -1 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
$$

If we put this LP problem in standard form:

$$
\begin{array}{rrcrcr}
\max & -2y_1 & + & 2y_2 & = & w \\
 & -y_1 & + & 2y_2 & \leq & -1 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
$$

and looking at the tableau:

y1	y2	s1	s2	-z	b
-1	2	1	0	0	-1
-1	2	0	1	0	1
-2	2	0	0	1	0

we see that the initial tableau like for the primal problem is infeasible.
However, the dual problem has an advantage, if we change temporarily the objective function of the primal problem to $\eta = -x_1 - x_2$, the dual problm becomes:

$$
\begin{array}{rl}
\max -x_1 - x_2 = \eta \\
x_1 + x_2 \leq 2 \\
2x_1 + 2x_2 \geq 2 \\
x_1, x_2 \geq 0
\end{array}
\qquad
\begin{array}{r}
\min 2y_1 - 2y_2 = \gamma \\
y_1 - 2y_2 \geq 1 \\
y_1 - 2y_2 \geq -1 \\
y_1, y_2 \geq 0
\end{array}
\qquad
\begin{array}{r}
\max -2y_1 + 2y_2 = \gamma \\
-y_1 + 2y_2 \leq 1 \\
-y_1 + 2y_2 \leq 1 \\
y_1, y_2 \geq 0
\end{array}
$$

and the corresponding tableau has an easy basic feasible solution:

y1	y2	s1	s2	-z	b
-1	2	1	0	0	1
-1	2	0	1	0	1
-2	2	0	0	1	0

We can then solve to optimality with the primal simplex: the variable y_2 enters the basis and the variable s_2 exits. The new tableau becomes:

	y1	y2	s1	s2	-z	b
R1	-1	2	1	0	0	1
R2'=R2/2	-1/2	1	0	1/2	0	1/2

R3'=R3-R2	-1	0	0	-1	1	-1/2

and it is optimal. We can do the same iteration on the primal but with the dual simplex. Let's write the tableau of the primal with the objective function temporarily changed and keeping the old objective as well:

x1	x2	s1	s2	-z	-e	b
1	1	1	0	0	0	2
-2	-2	0	1	0	0	-2
1	-1	0	0	1	0	0
-1	-1	0	0	0	1	0

As we see we have the conditions of the dual simplex satisfied, the tableau is optimal but not feasible. Let's make an iteration of the dual simplex. We choose the row with negative b term and the column with negative pivot that minimizes the ratio test: $|c/a|$. We choose the second row and the second column (again watch out that we do not consider the row of the addd old objective to decide the row). In other terms we try to make the solution feasible while minimizing the loss in quality. The opertations to update the tableau remain the same as for the primal simplex. We obtan:

	x1	x2	s1	s2	-z	-e	b
R1'=R1-R2'	0	0	1	1/2	0	0	1
R2'=-1/2R2	1	1	0	-1/2	0	0	1
R3'=R3+R2'	2	0	0	-1/2	1	0	1
R4'=R4+R2'	0	0	0	-1/2	0	1	1

This tableau is optimal for the dual simplex, this means that a feasible solution for the primal problem has been found: $(0, 1, 1, 0)$. We can now proceed with the primal simplex.
Note that the considerations on the dual problem made above were just for explanation purposes, when solving our LP problem we do not need to write down the dual form of it or its tableaux. Instead, we just need to switch from dual simplex to primal simplex always working on the original (the primal) formulation of the problem. The dual simplex method simply a new way of picking the entering and leaving variables in a sequence of primal tableaux.

Phase II We can now remove the temporary objective function and the corresponding column and proceed with the primal simplex.

x1	x2	s1	s2	-z	b
0	0	1	1/2	0	1
1	1	0	-1/2	0	1
2	0	0	-1/2	1	1

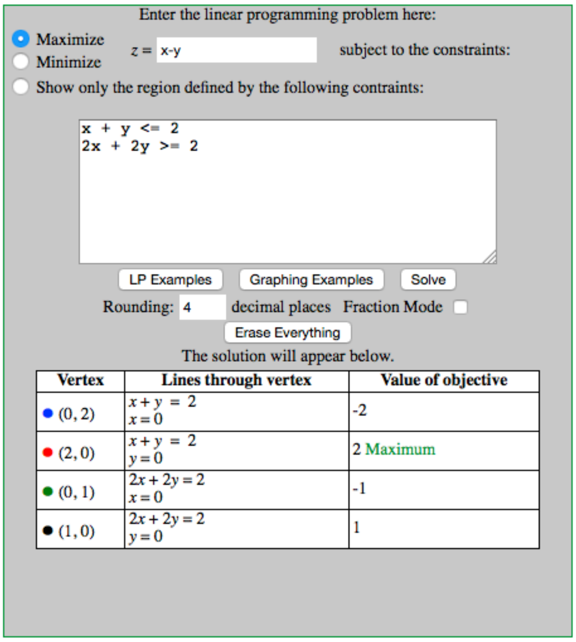
x_1 enters the basis and x_2 exits. The tableau is updated consequently:

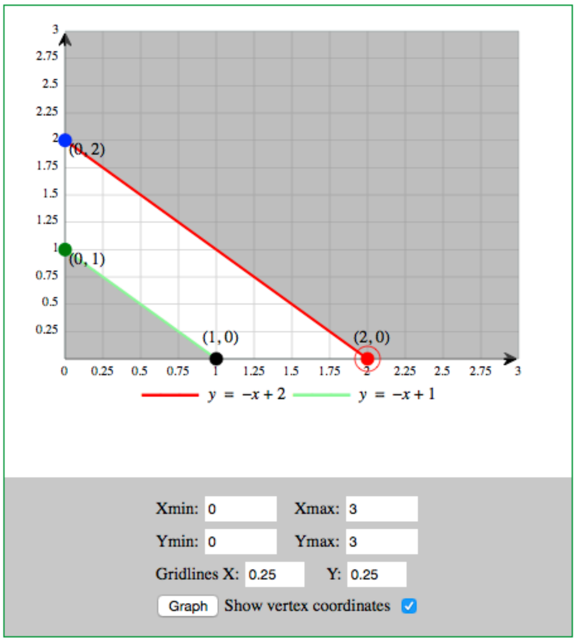
```
|              | x1  | x2  | s1  | s2    | -z  |  b  |
|--------------+-----+-----+-----+-------+-----+-----|
| R1'=R1       |  0  |  0  |  1  | 1/2   |  0  |  1  |
| R2'=R2       |  1  |  1  |  0  | -1/2  |  0  |  1  |
| R3'=R3-2*R2  |  0  | -2  |  0  | 1/2   |  1  | -1  |
|--------------+-----+-----+-----+-------+-----+-----|
```

A reduced cost is still positive, hence we make s_2 enters in the basis and s_1 leave. This leads to

```
|            | x1 | x2 | s1 | s2 | -z |  b |
|------------+----+----+----+----+----+----|
| R1'=2*R1   |  0 |  0 |  2 |  1 |  0 |  2 |
| R2'=R2+R1  |  1 |  1 |  1 |  0 |  0 |  2 |
| R3'=R3-R1  |  0 | -2 | -1 |  0 |  1 | -2 |
|------------+----+----+----+----+----+----|
```

The tableau is now optimal and the corresponding basic feasible solution is $\mathbf{x} = (2, 0)$ and has value $z^* = 2$.
We can visualize the problem using the LP Grapher tool linked from the course webpage:





Exercise 5

For each of the following LPs, express the optimal value and the optimal solution in terms of the problem parameters c. If the optimal solution is not unique, it is sufficient to give one optimal solution.

(a)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}. \end{aligned}$$

The variable is $\mathbf{x} \in \mathbb{R}^n$

(b)

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & -\mathbf{1} \leq \mathbf{x} \leq \mathbf{1}. \end{aligned}$$

The variable is $\mathbf{x} \in \mathbb{R}^n$.

Exercise 6* Quzzies

Basic Geometric Facts

1. In 4D, how many hyperplanes need to intersect to give a point?

   Solution:

   4

2. In 4D, can a point be described by more than 4 hyperplanes?

   Solution:

   Yes, just think of a pyramid in 3D

3. Consider the intersection of n hyperplanes in n dimensions: when does it uniquely identify a point?

   Solution:

   when the rank of the matrix A of the linear system is n (or A is nonsingular)

Vertices of Polyhedra:
Consider the polyhedron described by $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$, that is:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\
 & & \vdots & & & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m
\end{array}$$

4. How many constraints are *active* in a *vertex* of a polyhedron $A\mathbf{x} \leq \mathbf{b}$, $A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^n$?

   Solution:

   at least n, rank of matrix of active constraints is n

5. Does every point x that activates n constraints form a vertex of the polyhedron?

   Solution:

   no, some may be not feasible, ie, intersection in a point outside of the polyhedron

6. Can a vertex activate more than n constraints?

   Solution:

   Yes, just look at the pyramid in 3 dim. However, the rank of the matrix of active constraints is still n

7. What if there are more variables than constraints? If $m > n$ then we can find a subset and then activate but what if $m < n$, can we have a vertex?

   Solution:

   No. In LP we deal with this issue by adding slack variables, they make us choose arbitrarily a vertex

8. Combinatorial explosion of vertices: how many constraints and vertices has an n-dimensional hypercube?

   Solution:

   To define a cube we need 6 constraints and there are 2^3 vertices. For an n-hypercube we need $2n$ constraints and there are 2^n vertices

9. If there are m constraints and n variables, $m > n$, what is an upper bound to the number of vertices?

   Solution:

   the number of possible active constraints is $\binom{m}{n}$ it is an upper bound because:

   – some combinations of constraints will not define a vertex, ie, if rows of matrix not independent

   – some vertices are outside the polyhedron

   – some vertices may activate more than n constraints and hence the same vertex can be given by more than n constraints

Tableaux and Vertices

10. For each of these three statements, say if they are true or false:

    – One tableau $\Longrightarrow$ one vertex of the feasible region

    – One tableau $\Longleftarrow$ one vertex of the feasible region

    – One tableau $\Longleftrightarrow$ one vertex of the feasible region

    Solution:

    One tableau $\not\Longleftarrow$ one vertex of the feasible region degenerate vertices have several tableau associated

11. Consider the following LP problem and the corresponding final tableau:

$$
\begin{array}{rrcrcl}
\max & 6x_1 & + & 8x_2 & & \\
 & 5x_1 & + & 10x_2 & \leq & 60 \\
 & 4x_1 & + & 4x_2 & \leq & 40 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

	x_1	x_2	x_3	x_4	$-z$	b
x_2	0	1	1/5	−1/4	0	2
x_1	1	0	−1/5	1/2	0	8
	0	0	−2/5	−1	1	−64

– How many variables (original and slack) can be different from zero?

Solution:

at most 2

– $(x_3, x_4) = (0, 0)$ are non basic, what does this tell us about the constraints?

Solution:

They are active because their dual values are not zero

Let's generalize the previous case. Consider an LP with m constraints, n original variables and m slack variables. In an optimal solution:

– is $m > n$, how many variables (original and slack) can be nonzero at most?

Solution:

at most m

– If $m < n$ how many original variables must be zero at least? In other terms, in a mix planning problem with n products and m, $m < n$ resources, how many products at most will be to be produced in an optimal solution?

Solution:

$n - m$, and hence at most $m < n$ products

Solution:

at most m

12. Consider the following LP problem and the corresponding final tableau:

$$\begin{array}{rrcrcr} \max & 6x_1 & + & 8x_2 & & \\ & 5x_1 & + & 10x_2 & \leq & 60 \\ & 4x_1 & + & 4x_2 & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

	x_1	x_2	x_3	x_4	$-z$	b
x_3	0	0	1	1/2	0	1
x_1	1	1	0	$-1/2$	0	1
	0	-2	0	1/2	1	-1

$(x_2, x_4) = (0, 0)$ is non basic, what does this tell us about the constraints?

Solution:

The second constraint is active because its slack x_4 is zero. $x_2 = 0 \implies x_2 \geq 0$ is active.

13. If in the original space of the problem we had 3 variables, and there are 6 constraints, how many constraints would be active?

Solution:

3 constraints. With slack variables we would have 6 variables in all, if any of them is positive the constraint $x_i \geq 0$ of the original variables would be active, otherwise the corresponding constraints of the original problem are active.

14. For the general case with n original variables:
One basic feasible solution $\iff$ a matrix of active constraints has rank n. True or False?

Solution:

True

15. Consider an LP problem with m constraints and n original variables, $m > n$. We saw that in $\mathbb{R}^n$ a point is the intersection of at least n hyperplanes. In LP this corresponds to say that in a vertex there are n active constraints. Let a tableau be associated with a solution that makes exactly $n + 1$ constraints active, what can we say about the corresponding basic and non-basic variable values?

Solution:

one basic variable is zero. Indeed, in the simplex we will have $m + n$ variables and m variables in basis. We saw that the n non basic variables are set to zero and that there is an active constraint for each of them. Hence, if there are $n + 1$ active constraints, there must be another variable that is set to zero. It must be a basic variable.

16. What is the algebraic definition of adjacency in 2, 3 and n dimensions?

Solution:

two vertices are adjacent iff:

– they have at least $n - 1$ active constraints in common
– rank of common active constraints is $n - 1$

17. How does this condition translate in terms of tableau?

Solution:

For what seen above this translates in $n - 1$ variables in common in the tableau