DM545

Linear and Integer Programming

Lecture 11
Network Flows

Marco Chiarandini

Department of Mathematics & Computer Science
University of Southern Denmark

Outline

1. (Minimum Cost) Network Flows

2. Duality in Network Flow Problems

3. Assignment and Transportation

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1. (Minimum Cost) Network Flows

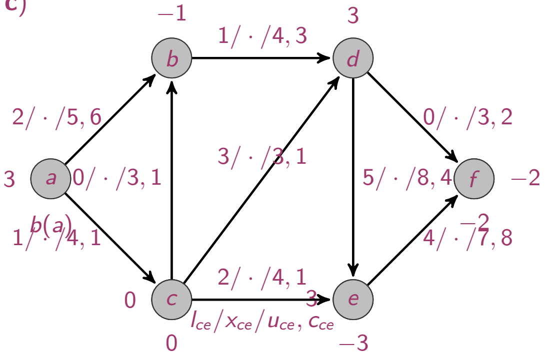
2. Duality in Network Flow Problems

3. Assignment and Transportation

Terminology

Network:
- directed graph $D = (V, A)$
- arc, directed link, from tail to head
- lower bound $l_{ij} > 0$, $\forall ij \in A$, capacity $u_{ij} \geq l_{ij}$, $\forall ij \in A$
- cost c_{ij}, linear variation (if $ij \notin A$ then $l_{ij} = u_{ij} = 0$, $c_{ij} = 0$)
- balance vector $b(i)$, $b(i) > 0$ supply node (source), $b(i) < 0$ demand node (sink, tank), $b(i) = 0$ transhipment node (assumption $\sum_i b(i) = 0$)

$N = (V, A, \mathbf{l}, \mathbf{u}, \mathbf{b}, \mathbf{c})$



Network Flows

Flow $\mathbf{x} : A \to \mathbb{R}$

balance vector of $\mathbf{x}$: $b_{\mathbf{x}}(v) = \sum_{vu \in A} x_{vu} - \sum_{wv \in A} x_{wv}, \forall v \in V$

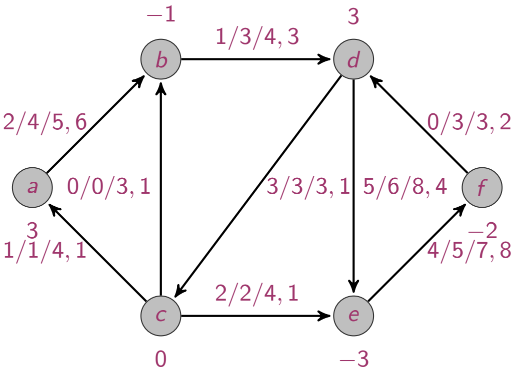
$$b_{\mathbf{x}}(v) \begin{cases} > 0 & \text{source} \\ < 0 & \text{sink/target/tank} \\ = 0 & \text{balanced} \end{cases}$$

(generalizes the concept of path with $b_{\mathbf{x}}(v) = \{0, 1, -1\}$)

feasible $\quad l_{ij} \le x_{ij} \le u_{ij}$, $b_{\mathbf{x}}(i) = b(i)$
cost $\quad \mathbf{c}^T\mathbf{x} = \sum_{ij \in A} c_{ij} x_{ij}$ (varies linearly with $\mathbf{x}$)

If iji is a 2-cycle and all $l_{ij} = 0$, then at least one of x_{ij} and x_{ji} is zero.

Example



Feasible flow of cost 109

Minimum Cost Network Flows

Find cheapest flow through a network in order to satisfy demands at certain nodes from available supplier nodes.

Variables:

$$x_{ij} \in \mathbb{R}_0^+$$

Objective:

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

Constraints: mass balance + flow bounds

$$\sum_{j:ij \in A} x_{ij} - \sum_{j:ji \in A} x_{ji} = b(i) \quad \forall i \in V$$

$$l_{ij} \leq x_{ij} \leq u_{ij}$$

$$\begin{aligned} \min\ & \mathbf{c}^T \mathbf{x} \\ & N\mathbf{x} = \mathbf{b} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

N node arc incidence matrix

(assumption: all values are integer, we can multiply if rational)

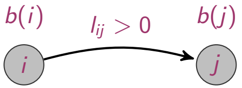
$$
\begin{array}{r:cccccccc}
 & x_{e_1} & x_{e_2} & \cdots & x_{ij} & \cdots & x_{e_m} & & \\
 & c_{e_1} & c_{e_2} & \cdots & c_{ij} & \cdots & c_{e_m} & & \\
\hdashline
1 & -1 & . & \cdots & . & \cdots & . & = & b_1 \\
2 & . & . & \cdots & . & \cdots & . & = & b_2 \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
i & 1 & . & \cdots & -1 & \cdots & . & = & b_i \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
j & . & . & \cdots & 1 & \cdots & . & = & b_j \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
n & . & . & \cdots & . & \cdots & . & = & b_n \\
\hdashline
e_1 & 1 & & & & & & \leq & u_1 \\
e_2 & & 1 & & & & & \leq & u_2 \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
(i,j) & & & & 1 & & & \leq & u_{ij} \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
e_m & & & & & & 1 & \leq & u_m
\end{array}
$$

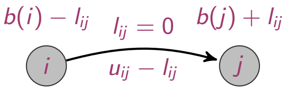
Reductions/Transformations

Lower bounds

Let $N = (V, A, \mathbf{l}, \mathbf{u}, \mathbf{b}, \mathbf{c})$

$$
\begin{aligned}
N' &= (V, A, \mathbf{l}', \mathbf{u}', \mathbf{b}', \mathbf{c}) \\
b'(i) &= b(i) - l_{ij} \\
b'(j) &= b(j) + l_{ij} \\
u'_{ij} &= u_{ij} - l_{ij} \\
l'_{ij} &= 0
\end{aligned}
$$





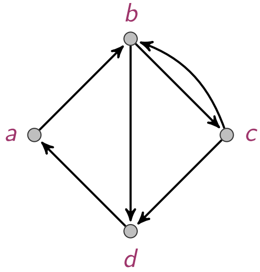
$$\mathbf{c}^T\mathbf{x}$$

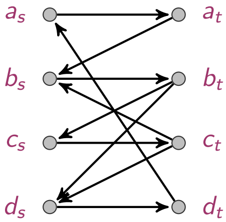
$$\mathbf{c}^T\mathbf{x}' + \sum_{ij \in A} c_{ij} l_{ij}$$

Undirected arcs

Vertex splitting

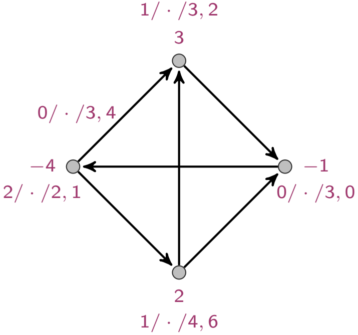
If there are bounds and costs of flow passing through vertices where $b(v) = 0$ (used to ensure that a node is visited):
$N = (V, A, \mathbf{l}, \mathbf{u}, \mathbf{c}, \mathbf{l}^*, \mathbf{u}^*, \mathbf{c}^*)$

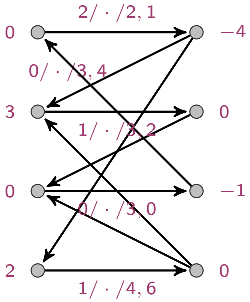




From D to D_{ST} as follows:

$$\begin{aligned} \forall v \in V \quad & \rightsquigarrow v_s, v_t \in V(D_{ST}) \text{ and } v_s v_t \in A(D_{ST}) \\ \forall xy \in A(D) & \rightsquigarrow x_t y_s \in A(D_{ST}) \end{aligned}$$



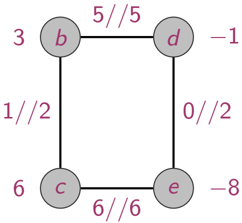


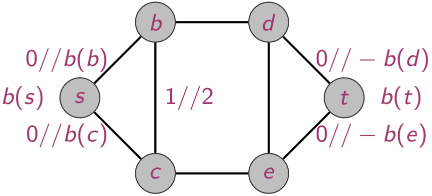
$$\forall v \in V \text{ and } v_s v_t \in A_{ST} \rightsquigarrow h'(v_s v_t) = h^*(v), \quad h^* \in \{l^*, u^*, c^*\}$$
$$\forall xy \in A \text{ and } x_t y_s \in A_{ST} \rightsquigarrow h'(x_t y_s) = h(x, y), \; h \in \{l, u, c\}$$

If $b(v) = 0$, then $b'(v_s) = b'(v_t) = 0$
If $b(v) < 0$, then $b'(v_s) = 0$ and $b'(v_t) = b(v)$
If $b(v) > 0$, then $b'(v_s) = b(v)$ and $b'(v_t) = 0$

(s,t)-flow:

$$b_{\mathbf{x}}(v) = \begin{cases} k & \textit{if } v = s \\ -k & \textit{if } v = t \\ 0 & \textit{otherwise} \end{cases}, \qquad |\mathbf{x}| = |b_{\mathbf{x}}(s)|$$





$$b(s) = \sum_{v:b(v)>0} b(v) = M$$
$$b(t) = \sum_{v:b(v)<0} b(v) = -M$$

$\exists$ feasible flow in N $\iff$ $\exists$ (s,t)-flow in N_{st} with $|x| = M$ $\iff$ max flow in N_{st} is M

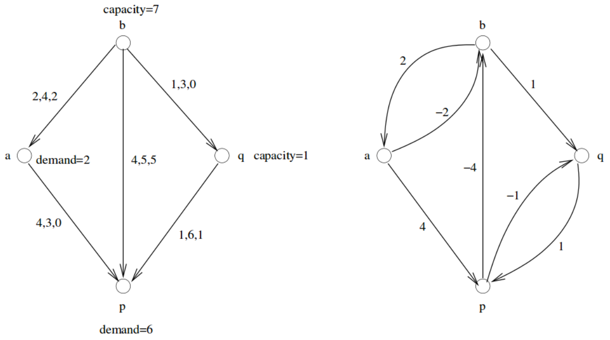
Residual Network

Residual Network $N(\mathbf{x})$: given that a flow $\mathbf{x}$ already exists, how much flow excess can be moved in G?

Replace arc $ij \in N$ with arcs:

	residual capacity	cost
ij :	$r_{ij} = u_{ij} - x_{ij}$	c_{ij}
ji :	$r_{ji} = x_{ij}$	$-c_{ij}$

$(N, \mathbf{c}, \mathbf{u}, \mathbf{x})$ $\qquad$ $(N(\mathbf{x}), \mathbf{c}')$



Special cases

Shortest path problem path of minimum cost from s to t with costs $\lesseqgtr 0$

$b(s) = 1, b(t) = -1, b(i) = 0$

if to any other node? $b(s) = (n-1), b(i) = 1, u_{ij} = n-1$

Max flow problem incur no cost but restricted by bounds

steady state flow from s to t

$b(i) = 0 \ \forall i \in V, \qquad c_{ij} = 0 \ \forall ij \in A \qquad ts \in A$

$c_{ts} = -1, \qquad u_{ts} = \infty$

Assignment problem min weighted bipartite matching,

$|V_1| = |V_2|, A \subseteq V_1 \times V_2$

c_{ij}

$b(i) = 1 \ \forall i \in V_1 \qquad b(i) = -1 \ \forall i \in V_2 \qquad u_{ij} = 1 \ \forall ij \in A$

Special cases

Transportation problem/Transshipment distribution of goods, warehouses-costumers

$|V_1| \neq |V_2|, \qquad u_{ij} = \infty$ for all $ij \in A$

$$\min \sum c_{ij} x_{ij}$$
$$\sum_i x_{ij} \geq b_j \qquad \forall j$$
$$\sum_j x_{ij} \leq a_i \qquad \forall i$$
$$x_{ij} \geq 0$$

if $\sum a_i = \sum b_i$ then $\geq / \leq$ become $=$
if $\sum a_i > \sum b_i$ then add dummy tank nodes
if $\sum a_i < \sum b_i$ then infeasible

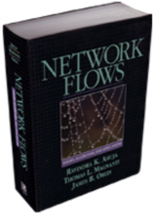
Multi-commodity flow problem ship several commodities using the same network, different origin destination pairs separate mass balance constraints, share capacity constraints, min overall flow

$$
\begin{aligned}
\min \textstyle\sum_k \mathbf{c}^k \mathbf{x}^k & \\
N\mathbf{x}^k \geq \mathbf{b}^k & \qquad \forall k \\
\textstyle\sum_k \mathbf{x}_{ij}^k \leq \mathbf{u}_{ij} & \qquad \forall ij \in A \\
0 \leq \mathbf{x}_{ij}^k \leq \mathbf{u}_{ij}^k &
\end{aligned}
$$

What is the structure of the matrix now? Is the matrix still TUM?

Application Example
Ship loading problem

Plenty of applications. See Ahuja Magnanti Orlin, Network Flows, 1993

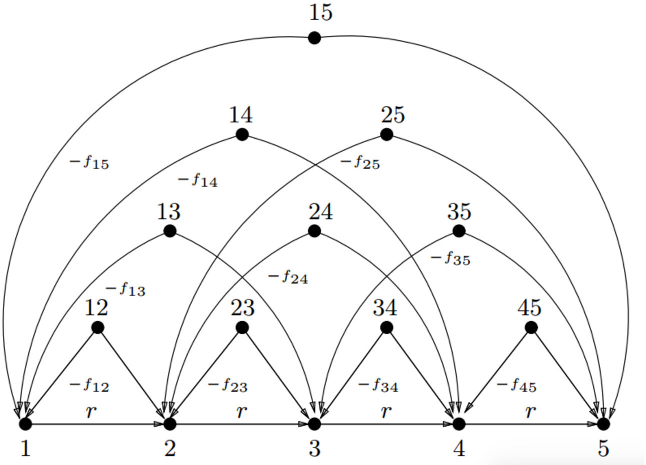


- A cargo company (eg, Maersk) uses a ship with a capacity to carry at most r units of cargo.
- The ship sails on a long route (say from Southampton to Alexandria) with several stops at ports in between.
- At these ports cargo may be unloaded and new cargo loaded.
- At each port there is an amount b_{ij} of cargo which is waiting to be shipped from port i to port $j > i$
- Let f_{ij} denote the income for the company from transporting one unit of cargo from port i to port j.
- The goal is to plan how much cargo to load at each port so as to maximize the total income while never exceeding ship's capacity.

Application Example: Modeling

- n number of stops including the starting port and the terminal port.
- $N = (V, A, \mathbf{l} \equiv \mathbf{0}, \mathbf{u}, \mathbf{c})$ be the network defined as follows:
  - $V = \{v_1, v_2, ..., v_n\} \cup \{v_{ij} : 1 \leq i < j \leq n\}$
  - $A = \{v_1 v_2, v_2 v_3, ... v_{n-1} v_n\} \cup \{v_{ij} v_i, v_{ij} v_j : 1 \leq i < j \leq n\}$
  - capacity: $u_{v_i v_{i+1}} = r$ for $i = 1, 2, ..., n-1$ and all other arcs have capacity ∞.
  - cost: $c_{v_{ij} v_i} = -f_{ij}$ for $1 \leq i < j \leq n$ and all other arcs have cost zero (including those of the form $v_{ij} v_j$)
  - balance vector: $b(v_{ij}) = b_{ij}$ for $1 \leq i < j \leq n$ and the balance vector of $b(v_i) = -b_{1i} - b_{2i} - ... - b_{i-1,i}$ for $i = 1, 2, ..., n$

Application Example: Modeling



Application Example: Modeling

Claim: the network models the ship loading problem.

- suppose that $t_{12}, t_{13}, \ldots, t_{1n}, t_{23}, \ldots, t_{n-1,n}$ are cargo numbers, where t_{ij} $(\leq b_{ij})$ is the amount of cargo the ship will transport from port i to port j and that the ship is never loaded above capacity.
- total income is

$$I = \sum_{1 \leq i < j \leq n} t_{ij} f_{ij}$$

- Let x be the flow in N defined as follows:
  - flow on an arc of the form $v_{ij}v_i$ is t_{ij}
  - flow on an arc of the form $v_{ij}v_j$ is $b_{ij} - t_{ij}$
  - flow on an arc of the form $v_i v_{i+1}$, $i = 1, 2, \ldots, n-1$, is the sum of those t_{ab} for which $a \leq i$ and $b \geq i + 1$.
- since t_{ij}, $1 \leq i < j \leq n$, are legal cargo numbers then x is feasible with respect to the balance vector and the capacity restriction.
- the cost of x is $-I$.

Application Example: Modeling

- Conversely, suppose that x is a feasible flow in N of cost J.
- we construct a feasible cargo assignment $s_{ij}, 1 \leq i < j \leq n$ as follows:
  - let s_{ij} be the value of x on the arc $v_{ij}v_i$.
- income $-J$

Outline

Shortest Path - Dual LP

$$z = \min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ji \in A} x_{ji} - \sum_{j:ij \in A} x_{ij} = 1 \qquad \textit{for } i = s \qquad (\pi_s)$$

$$\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = 0 \qquad \forall i \in V \setminus \{s, t\} \qquad (\pi_i)$$

$$\sum_{j:ji \in A} x_{ji} - \sum_{j:ij \in A} x_{ij} = -1 \qquad \textit{for } i = t \qquad (\pi_t)$$

$$x_{ij} \geq 0 \qquad \forall ij \in A$$

Dual problem:

$$g^{LP} = \max \pi_s - \pi_t$$

$$\pi_j - \pi_i \leq c_{ij} \qquad \forall ij \in A$$

Hence, the shortest path can be found by potential values π_i on nodes such that $\pi_s = z, \pi_t = 0$ and $\pi_j - \pi_i \leq c_{ij}$ for $ij \in A$

Maximum (s,t)-Flow

Adding a backward arc from t to s:

$$
\begin{aligned}
z = \max x_{ts} & & \\
\sum_{j:ji\in A} x_{ij} - \sum_{j:ij\in A} x_{ji} = 0 & \qquad \forall i \in V & \qquad (\pi_i) \\
x_{ij} \leq u_{ij} & \qquad \forall ij \in A & \qquad (w_{ij}) \\
x_{ij} \geq 0 & \qquad \forall ij \in A &
\end{aligned}
$$

Dual problem:

$$
\begin{aligned}
g^{LP} = \min \sum_{ij\in A} u_{ij} w_{ij} & \\
\pi_i - \pi_j + w_{ij} \geq 0 & \qquad \forall ij \in A \\
\pi_t - \pi_s \geq 1 & \\
w_{ij} \geq 0 & \qquad \forall ij \in A
\end{aligned}
$$

$$
\begin{array}{r|cccccccc}
 & x_{e_1} & x_{e_2} & \cdots & x_{ij} & \cdots & x_{e_m} & & \\
 & c_{e_1} & c_{e_2} & \cdots & c_{ij} & \cdots & c_{e_m} & & \\
\hline
1 & -1 & . & \cdots & . & \cdots & . & = & b_1 \\
2 & . & . & \cdots & . & \cdots & . & = & b_2 \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
i & 1 & . & \cdots & -1 & \cdots & . & = & b_i \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
j & . & . & \cdots & 1 & \cdots & . & = & b_j \\
\vdots & \vdots & \ddots & & & & & = & \vdots \\
n & . & . & \cdots & . & \cdots & . & = & b_n \\
\hline
e_1 & 1 & & & & & & \leq & u_1 \\
e_2 & & 1 & & & & & \leq & u_2 \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
(i,j) & & & & 1 & & & \leq & u_{ij} \\
\vdots & \vdots & \ddots & & & & & \leq & \vdots \\
e_m & & & & & & 1 & \leq & u_m
\end{array}
$$

$$
\begin{aligned}
g^{LP} &= \min \sum_{ij \in A} u_{ij} w_{ij} && && (1)\\
\pi_i - \pi_j + w_{ij} &\geq 0 && \forall ij \in A && (2)\\
\pi_t - \pi_s &\geq 1 && && (3)\\
w_{ij} &\geq 0 && \forall ij \in A && (4)
\end{aligned}
$$

- Without (3) all potentials would go to 0.
- Keep w low because of objective function
- Keep all potentials low $\rightsquigarrow$ (3) $\pi_s = 0, \pi_t = 1$
- Cut C: on left =1 on right =0. Where is the transition?
- Vars w identify the cut $\rightsquigarrow$ $\pi_j - \pi_i + w_{ij} \geq 0 \rightsquigarrow w_{ij} = 1$

  $$
  w_{ij} = \begin{cases} 1 & if \;\; ij \in C \\ 0 & otherwise \end{cases}
  $$

  for those arcs that minimize the cut capacity $\sum_{ij \in A} u_{ij} w_{ij}$
- Complementary slackness: $w_{ij} = 1 \implies x_{ij} = u_{ij}$

Theorem

A strong dual to the max (st)-flow is the minimum (st)-cut problem:

$$\min_{X} \left\{ \sum_{ij \in A : i \in X, j \notin X} u_{ij} : s \in X \subset V \setminus \{t\} \right\}$$

Max Flow Algorithms

Optimality Condition

- Ford Fulkerson augmenting path algorithm $O(m|x^*|)$
- Edmonds-Karp algorithm (augment by shortest path) in $O(nm^2)$
- Dinic algorithm in layered networks $O(n^2m)$
- Karzanov's push relabel $O(n^2m)$

Min Cost Flow - Dual LP

$$\min \sum_{ij \in A} c_{ij} x_{ij}$$

$$\sum_{j:ji \in A} x_{ij} - \sum_{j:ij \in A} x_{ji} = b_i \qquad \forall i \in V \qquad (\pi_i)$$

$$x_{ij} \leq u_{ij} \qquad \forall ij \in A \qquad (w_{ij})$$

$$x_{ij} \geq 0 \qquad \forall ij \in A$$

Dual problem:

$$\max \sum_{i \in V} b_i \pi_i - \sum_{ij \in E} u_{ij} w_{ij} \tag{1}$$

$$-c_{ij} - \pi_i + \pi_j \leq w_{ij} \qquad \forall ij \in E \tag{2}$$

$$w_{ij} \geq 0 \qquad \forall ij \in A \tag{3}$$

- define reduced costs $\bar{c}_{ij} = c_{ij} + \pi_j - \pi_i$, hence (2) becomes $-\bar{c}_{ij} \leq w_{ij}$
- $u_e = \infty$ then $w_e = 0$ (from obj. func) and $\bar{c}_{ij} \geq 0$ (optimality condition)
- $u_e < \infty$ then $w_e \geq 0$ and $w_e \geq -\bar{c}_{ij}$ then $w_e = \max\{0, -\bar{c}_{ij}\}$, hence w_e is determined by others and irrelevant
- Complementary slackness th. for optimal solutions:
  each primal variable $\times$ the corresponding dual slack must be equal 0, ie, $x_e(\bar{c}_e + w_e) = 0$;
  - $x_e > 0$ then $-\bar{c}_e = w_e = \max\{0, -\bar{c}_e\}$,
    $x_e > 0 \implies -\bar{c}_e \geq 0$ or equivalently (by negation) $\bar{c}_e > 0 \implies x_e = 0$

  each dual variable $\times$ the corresponding primal slack must be equal 0, ie, $w_e(x_e - u_e) = 0$;
  - $w_e > 0$ then $x_e = u_e$
    $-\bar{c} > 0 \implies x_e = u_e$ or equivalently $\bar{c} < 0 \implies x_e = u_e$

  Hence:
  $\bar{c}_e > 0$ then $x_e = 0$
  $\bar{c}_e < 0$ then $x_e = u_e \neq \infty$

Min Cost Flow Algorithms

Theorem (Optimality conditions)

Let $\mathbf{x}$ be feasible flow in $N(V, A, \mathbf{l}, \mathbf{u}, \mathbf{b})$ then $\mathbf{x}$ is min cost flow in N iff $N(\mathbf{x})$ contains no directed cycle of negative cost.

- Cycle canceling algorithm with Bellman Ford Moore for negative cycles $O(nm^2UC)$, $U = \max|u_e|$, $C = \max|c_e|$
- Build up algorithms $O(n^2mM)$, $M = \max|b(v)|$

Outline

Assignment Problem

Input: a set of persons $P_1, P_2, \ldots, P_n$, a set of jobs $J_1, J_2, \ldots, J_n$ and an $n \times n$ matrix $M = [M_{ij}]$ whose entries are non-negative integers. Here M_{ij} is a measure for the skill of person P_i in performing job J_j (the lower the number the better P_i performs job J_j).
Goal is to find an assignment π of persons to jobs so that each person gets exactly one job and the sum $\sum_{i=1}^{n} M_{i\pi(i)}$ is minimized.

Matching Algorithms

Matching: $M \subseteq E$ of pairwise non adjacent edges

- bipartite graphs
- arbitrary graphs

- cardinality (max or perfect)
- weighted

Assignment problem $\equiv$ min weighted perfect bipartite matching $\equiv$ special case of min cost flow

bipartite cardinality

Theorem

The cardinality of a max matching in a bipartite graph equals the value of a maximum (s,t)-flow in N_{st}.

$\rightsquigarrow$ Dinic $O(\sqrt{n}m)$

Theorem (Optimality condition (Berge))

A matching M in a graph G is a maximum matching iff G contains no M-augmenting path.

$\rightsquigarrow$ augmenting path $O(\min(|U|,|V|),m)$

bipartite weighted
build up algorithm $O(n^3)$
bipartite weighted: Hungarian method $O(n^3)$

minimum weight perfect matching
Edmonds $O(n^3)$

Theorem (Hall's (marriage) theorem)

A bipartite graph $B = (X, Y, E)$ *has a matching covering* X *iff:*

$$|N(U)| \geq |U| \quad \forall U \subseteq X$$

Theorem (König, Egeavary theorem)

Let $B = (X, Y, E)$ *be a bipartite graph. Let* M^* *be the maximum matching and* V^* *the minimum vertex cover:*

$$|M^*| = |V^*|$$

Transportation Problem

Given: a set of production plants $S_1, S_2, ..., S_m$ that produce a certain product to be shipped to a set of re-tailers $T_1, T_2, ..., T_n$. For each pair (Si, Tj) there is a real-valued cost c_{ij} of transporting one unit of the product from S_i to T_j. Each plant produces $a_i, i = 1, 2, ..., m$, units per time unit and each retailer needs $b_j, j = 1, 2, ..., n$, units of the product per time unit.

Goal: find a transportation schedule for the whole production (i.e., how many units to send from S_i to T_j for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$) in order to minimize the total transportation cost.

We assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Summary

1. (Minimum Cost) Network Flows

2. Duality in Network Flow Problems

3. Assignment and Transportation