DM559

Linear and Integer Programming

Lecture 3

Matrix Operations

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Outline

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Matrices

Definition (Matrix)

A matrix is a rectangular array of numbers or symbols. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- We denote this array by a single letter A or by (a_{ij}) and
- we say that A has m rows and n columns, or that it is an $m \times n$ matrix.
- The size of A is $m \times n$.
- The number a_{ij} is called the (i, j) entry or scalar.

- A square matrix is an $n \times n$ matrix.

- The diagonal of a square matrix is the list of entries $a_{11}, a_{22}, \ldots, a_{nn}$

- The diagonal matrix is a matrix $n \times n$ with $a_{ij} = 0$ if $i \neq j$ (ie, a square matrix with all the entries which are not on the diagonal equal to 0):

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mn} \end{bmatrix}$$

Definition (Equality)

Two matrices are equal if they have the same size and if corresponding entries are equal. That is, if $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then:

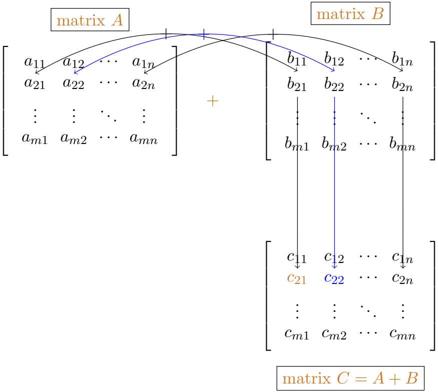
$$A = B \iff a_{ij} = b_{ij} \qquad 1 \leq i \leq m, 1 \leq j \leq n$$

Matrix Addition

Definition (Addition)

If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then

$$A + B = (a_{ij} + b_{ij}) \qquad 1 \leq i \leq m, 1 \leq j \leq n$$



Eg:

$$A + B = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 4 \\ 2 & -3 & 1 \end{bmatrix} = ?$$

element-wise operation

Scalar Matrix Multiplication

Definition (Scalar Multiplication)

If $A = (a_{ij})$ is an $m \times n$ matrix and $\lambda \in \mathbb{R}$, then

$$\lambda A = (\lambda a_{ij}) \qquad 1 \leq i \leq m, 1 \leq j \leq n$$

Eg:

$$-2A = ?$$

element-wise operation

Matrix Multiplication

Two matrices can be multiplied together, depending on the size of the matrices

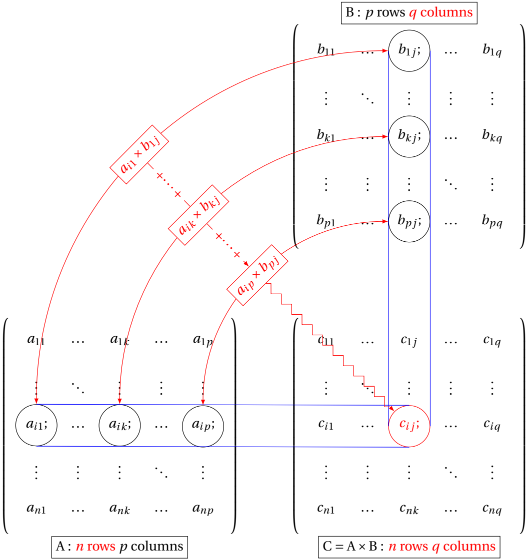
Definition (Matrix Multiplication)

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product is the matrix $AB = C = (c_{ij})$ with

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} & & & \\ & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix}$$

What is the size of C?



Not an element-wise operation!

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 4 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 3 \\ 1 & 14 \\ 9 & -1 \end{bmatrix}$$

$$(2)(3) + (0)(1) + (1)(-1) = 5$$

The motivation behind this definition is that it allows to deal conveniently with several tasks in linear algebra. Think about the way we rewrote a system of linear equations using this definition.

- $AB \neq BA$ in general, ie, not commutative
  try with the example of previous slide...

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad AB \text{ is } 2 \times 2 \text{ and } BA \text{ is } 3 \times 3$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{ok sizes but } AB \neq BA$$

Matrix Algebra

Matrices are useful because they provide compact notation and we can perform algebra with them

Bear in mind to use only operations that are defined. In the following rules, the sizes are dictated by the operations being defined.

- commutative $A + B = B + A$. Proof?
- associative:
  - $(A + B) + C = A + (B + C)$
  - $\lambda(AB) = (\lambda A)B = A(\lambda B)$
  - $(AB)C = A(BC)$

Size?

- distributive:
  - $A(B + C) = AB + AC$
  - $(B + C)A = BA + CA$
  - $\lambda(A + B) = \lambda A + \lambda B$

Why both first two rules?

Zero Matrix

Definition (Zero Matrix)

A zero matrix, denoted 0, is an $m \times n$ matrix with all entries zero:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

- additive identity: $A - A = 0$
    - $A + 0 = A$
    - $A - A = 0$
    - $0A = 0, A0 = 0$

Identity Matrix

Definition (Identity Matrix)

The $n \times n$ identity matrix, denoted I_n or I is the diagonal matrix with $a_{ii} = 1$: zero:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- multiplicative identity (like 1 does for scalars)

- $AI = A$ and $IA = A$

A of size $m \times n$.
What size is I?
$\rightsquigarrow$ the identity matrix must be a square matrix

Exercise: $3A + 2B = 2(B - A + C)$

Matrix Inverse

- If $AB = AC$ can we conclude that $B = C$?

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 8 & 0 \\ -4 & 4 \end{bmatrix}$$

$$AB = AC = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}$$

  but hold on, this might be just a lucky case

- $A + 5B = A + 5C \implies B = C$
  addition and scalar multiplication have inverses ($-A$ and $1/c$)

- Is there a multiplicative inverse?

Inverse Matrix

Definition (Inverse Matrix)

The $n \times n$ matrix A is invertible if there is a matrix B such that

$$AB = BA = I$$

where I is the $n \times n$ identity matrix. The matrix B is called the inverse of A and is denoted by A^{-1}.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Theorem

If A is an $n \times n$ invertible matrix, then the matrix A^{-1} is unique.

Proof: Assume A has two inverses B, C so $AB = BA = I$ and $AC = CA = I$. Consider the product CAB:

$$CAB = C(AB) = CI = C \qquad \text{associativity} + AB = I$$
$$CAB = (CA)B = IB = B \qquad \text{associativity} + CA = I$$

- If a matrix has an inverse we say that it is invertible or non-singular
  If a matrix has no inverse we say that it is non-invertible or singular
  Eg:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc \neq 0$$

  then A has the inverse

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad ad - bc \neq 0 \qquad \text{check that this is true}$$

- The scalar $ad - bc$ is called determinant of A and denoted $|A|$.

Matrix Inverse

Back to the question:

- If $AB = AC$ can we conclude that $B = C$?
  If A is invertible then the answer is yes:

$$A^{-1}AB = A^{-1}AC \implies IB = IC \implies B = C$$

- But $AB = CA$ then we cannot conclude that $B = C$.
  Note: the operation of matrix division is not defined!

Properties of the Inverse

Let A be invertible $\implies$ A^{-1} exists

- $(A^{-1})^{-1} = A$

- $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$
  the inverse of the matrix (λA) is a matrix C that satisfies $(\lambda A)C = C(\lambda A) = I$.
  Using matrix algebra:

$$(\lambda A)\left(\frac{1}{\lambda}A^{-1}\right) = \lambda\frac{1}{\lambda}AA^{-1} = I \text{ and } \left(\frac{1}{\lambda}A^{-1}\right)(\lambda A) = \frac{1}{\lambda}\lambda A^{-1}A = I$$

- $(AB)^{-1} = B^{-1}A^{-1}$

Powers of a matrix

For A an $n \times n$ matrix and $r \in \mathbb{N}$

$$A^r = \underbrace{AA \dots A}_{r \text{ times}}$$

For the associativity of matrix multiplication:

- $(A^r)^{-1} = (A^{-1})^r$
- $A^r A^s = A^{r+s}$
- $(A^r)^s = A^{rs}$

Transpose Matrix

Definition (Transpose)

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix B defined by

$$b_{ij} = a_{ji} \qquad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m$$

It is denoted A^T

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad A^T = (a_{ji}) = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nm} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

We reflect the matrix about its main diagonal

Note that if D is a diagonal matrix: $D^T = D$

Properties of the transpose

- $(A^T)^T = A$
- $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$ (consider first which matrix sizes make sense in the multiplication, then rewrite the terms)
- if A is invertible, $(A^T)^{-1} = (A^{-1})^T$

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I \qquad \text{using } (AB)^T = B^T A^T$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I \qquad \text{using } (AB)^T = B^T A^T$$

Symmetric Matrix

Definition (Symmetric Matrix)

A matrix A is symmetric if it is equal to its transpose, $A = A^T$.
(only square matrices can be symmetric)

Outline

Vectors

- An $n \times 1$ matrix is a column vector, or simply a vector:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

  The numbers $v_1, v_2, \ldots$ are known as the components (or entries) of v.

- A row vector is a $1 \times n$ matrix
- We write vectors in lower boldcase type (writing by hand we can either underline them or add an arrow over $\mathbf{v}$).
- Addition and scalar multiplication are defined for vectors as for $n \times 1$ matrices:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix} \qquad \lambda\mathbf{v} = \begin{bmatrix} \lambda v \\ \lambda v \\ \vdots \\ \lambda v \end{bmatrix}$$

element-wise operations

- For a fixed n, the set of vectors together with the operations of addition and multiplication form the set $\mathbb{R}^n$, usually called Euclidean space

- For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ in $\mathbb{R}$, the vector

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$$

  is known as linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$

- A zero vector is denoted by $\mathbf{0}$;
  $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$;
  $0\mathbf{v} = \mathbf{0}$

- The matrix product of $\mathbf{v}$ and $\mathbf{w}$ cannot be calculated
- The matrix product of $\mathbf{v}^T\mathbf{w}$ gives an 1×1 matrix
- The matrix product of $\mathbf{v}\mathbf{w}^T$ gives an $n \times n$ matrix

Inner product of two vectors

Definition (Inner product)

Given

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

the inner product denoted $\langle \mathbf{v}, \mathbf{w} \rangle$, is the real number given by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \left\langle \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \right\rangle = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n = \mathbf{v}^T \mathbf{w}$$

It is also called scalar product or dot product (and written $\mathbf{v} \cdot \mathbf{w}$).

$$\mathbf{v}^T\mathbf{w} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

Theorem

The inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

satisfies the following properties for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and for all* $\alpha \in \mathbb{R}$*:*

- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
- $\alpha \langle \mathbf{x}, \mathbf{y} \rangle = \langle \alpha\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \alpha\mathbf{y} \rangle$
- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$

Note: vectors from different Euclidean spaces live in different 'worlds'

Outline

1. Matrices

2. Vectors

3. Vectors and Matrices

4. More on Linear Systems

Vectors and Matrices

Let A be an $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and denote the columns of A by the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, so that

$$\mathbf{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, \qquad i = 1, \ldots, n.$$

Then if $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is any vector in $\mathbb{R}^n$

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

(ie, vector $A\mathbf{x}$ in $\mathbb{R}^m$ is a linear combination of the column vectors of A)

Resume

- We saw Matrix Algebra
- We can now prove two theorems on linear systems

Outline

Solution Sets of Linear Systems

Theorem

A system of linear equations either has no solution, a unique solution or infinitely many solutions.

Proof.
Let's assume the system $A\mathbf{x} = \mathbf{b}$ has two distinct solutions $\mathbf{p}$ and $\mathbf{q}$, that is:

$$A\mathbf{p} = \mathbf{b} \qquad \mathrm{A}\mathbf{q}{=}\mathbf{b} \qquad \mathbf{p} - \mathbf{q} \neq \mathbf{0}$$

Let t be any scalar and

$$\mathbf{v} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}), \quad t \in \mathbb{R}$$

Then:

$$A\mathbf{v} = A(\mathbf{p} + t(\mathbf{q} - \mathbf{p})) = A\mathbf{p} + t(A\mathbf{q} - A\mathbf{p}) = \mathbf{b} + t(\mathbf{b} - \mathbf{b}) = \mathbf{b}$$

that is, $\mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$ and since $\mathbf{p} - \mathbf{q} \neq \mathbf{0}$ and there are infinitely many choices for t, then there are infinitely many solutions for $A\mathbf{x} = \mathbf{b}$.

Theorem (Principle of Linearity)

Suppose that A is an $m \times n$ matrix, that $\mathbf{b} \in \mathbb{R}^m$ and that the system $A\mathbf{x} = \mathbf{b}$ is consistent.
Suppose that $\mathbf{p}$ is any solution of $A\mathbf{x} = \mathbf{b}$.
Then the set of all solutions of $A\mathbf{x} = \mathbf{b}$ consists precisely of the vectors $\mathbf{p} + \mathbf{z}$ for $\mathbf{z} \in N(A)$; ie,

$$\{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\} = \{\mathbf{p} + \mathbf{z} \mid \mathbf{z} \in N(A)\}.$$

Proof: We show that

1. $\mathbf{p} + \mathbf{z}$ is a solution for any $\mathbf{z}$ in the null space of A ($\{\mathbf{p} + \mathbf{z} \mid \mathbf{z} \in N(A)\} \subseteq \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\}$)
2. all solutions, $\mathbf{x}$, of $A\mathbf{x} = \mathbf{b}$ are of the form $\mathbf{p} + \mathbf{z}$ for some $\mathbf{z} \in N(A)$ ($\{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\} \subseteq \{\mathbf{p} + \mathbf{z} \mid \mathbf{z} \in N(A)\}$)

1. $A(\mathbf{p} + \mathbf{z}) = A\mathbf{p} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$ so $\mathbf{p} + \mathbf{z} \in \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}\}$

2. Let $\mathbf{x}$ be a solution. Because $\mathbf{p}$ is also we have $A\mathbf{p} = \mathbf{b}$ and$A(\mathbf{x} - \mathbf{p}) = A\mathbf{x} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$ so $\mathbf{z} = \mathbf{x} - \mathbf{p}$ is a solution of $A\mathbf{z} = \mathbf{0}$ and $\mathbf{x} = \mathbf{p} + \mathbf{z}$

(Check validity of the theorem on the last examples of previous lecture.)