DM559

Linear and Integer Programming

Lecture 9

Linear Transformations

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Outline

Resume

- vector spaces and subspaces
- range and null space, and rank
- linear independency
- bases and dimensions
- change of basis from standard to arbitrary basis
- change of basis between two arbitrary bases

Outline

Linear Transformations

Definition (Linear Transformation)

Let V and W be two vector spaces. A function $T : V \to W$ is linear if for all $\mathbf{u}, \mathbf{v} \in V$ and all $\alpha \in \mathbb{R}$:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$

A linear transformation is a linear function between two vector spaces

- If $V = W$ also known as linear operator
- Equivalent condition: $T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v})$
- for all $\mathbf{0} \in V$, $T(\mathbf{0}) = \mathbf{0}$

Example (Linear Transformations)

- vector space $V = \mathbb{R}$, $F_1(x) = px$ for any $p \in \mathbb{R}$

$$\begin{aligned}\forall x, y \in \mathbb{R}, \alpha, \beta \in \mathbb{R} : F_1(\alpha x + \beta y) = p(\alpha x + \beta y) &= \alpha(px) + \beta(py) \\ &= \alpha F_1(x) + \beta F_1(y)\end{aligned}$$

- vector space $V = \mathbb{R}$, $F_2(x) = px + q$ for any $p, q \in \mathbb{R}$ or $F_3(x) = x^2$ are not linear transformations

$$T(x+y) \neq T(x) + T(y) \quad \text{for some } x, y \in \mathbb{R}$$

- vector spaces $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $m \times n$ matrix A, $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$

$$\begin{aligned}&T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}) \\ &T(\alpha\mathbf{u}) = A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha T(\mathbf{u})\end{aligned}$$

Example (Linear Transformations)

- vector spaces $V = \mathbb{R}^n$, $W : f : \mathbb{R} \to \mathbb{R}$. $T : \mathbb{R}^n \to W$:

$$T(\mathbf{u}) = T\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) = p_{u_\mathbf{1}, u_\mathbf{2}, \dots, u_n} = p_\mathbf{u}$$

$$p_{u_\mathbf{1}, u_\mathbf{2}, \dots, u_n} = u_1 x^1 + u_2 x^2 + u_3 x^3 + \cdots + u_n x^n$$

$$\begin{aligned} p_{\mathbf{u}+\mathbf{v}}(x) &= \cdots = (p_\mathbf{u} + p_\mathbf{v})(x) \\ p_{\alpha\mathbf{u}}(\mathbf{x}) &= \cdots = \alpha p_u(x) \end{aligned}$$

Linear Transformations and Matrices

- any $m \times n$ matrix A defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow T_A$
- for every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is a matrix A such that $T(\mathbf{v}) = A\mathbf{v} \rightsquigarrow A_T$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ denote the standard basis of $\mathbb{R}^n$ and let A be the matrix whose columns are the vectors $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$: that is,

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{bmatrix}$$

Then, for every $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$.

Proof: write any vector $\mathbf{x} \in \mathbb{R}^n$ as lin. comb. of standard basis and then make the image of it.

Example

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x-y \\ x+2y-3z \end{bmatrix}$$

- The image of $\mathbf{u} = [1,2,3]^T$ can be found by substitution: $T(\mathbf{u}) = [6,-1,-4]^T$.
- to find A_T:

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_n)] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -3 \end{bmatrix}$$

$T(\mathbf{u}) = A\mathbf{u} = [6,-1,-4]^T$.

Linear Transformation in $\mathbb{R}^2$

- We can visualize them!

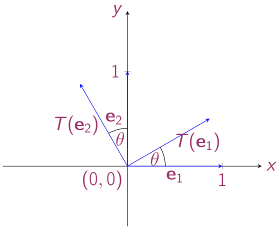
- Reflection in the x axis:

$$T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix} \qquad A_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Stretching the plane away from the origin

$$T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Rotation anticlockwise by an angle θ



we search the images of the standard basis vector $\mathbf{e}_1, \mathbf{e}_2$

$$T(\mathbf{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} b \\ d \end{bmatrix}$$

they will be orthogonal and with length 1.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

For $\pi/4$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Identity and Zero Linear Transformations

- For $T : V \to V$ the linear transformation such that $T(\mathbf{v}) = \mathbf{v}$ is called the identity.
- if $V = \mathbb{R}^n$, the matrix $A_T = I$ (of size $n \times n$)
- For $T : V \to W$ the linear transformation such that $T(\mathbf{v}) = \mathbf{0}$ is called the zero transformation.
- If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, the matrix A_T is an $m \times n$ matrix of zeros.

Composition of Linear Transformations

- Let $T : V \to W$ and $S : W \to U$ be linear transformations.
  The composition of ST is again a linear transformation given by:

  $$ST(\mathbf{v}) = S(T(\mathbf{v})) = S(\mathbf{w}) = \mathbf{u}$$

  where $\mathbf{w} = T(\mathbf{v})$

- ST means do T and then do S: $V \xrightarrow{T} W \xrightarrow{S} U$

- if $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ in terms of matrices:

  $$ST(\mathbf{v}) = S(T(\mathbf{v})) = S(A_T\mathbf{v}) = A_S A_T \mathbf{v}$$

  note that composition is not commutative

Combinations of Linear Transformations

- If $S, T : V \to W$ are linear transformations between the same vector spaces, then $S + T$ and αS, $\alpha \in \mathbb{R}$ are linear transformations.

- hence also $\alpha S + \beta T$, $\alpha, \beta \in \mathbb{R}$ is

Inverse Linear Transformations

- If V and W are finite-dimensional vector spaces of the same dimension, then the inverse of a lin. transf. $T : V \to W$ is the lin. transf such that

$$T^{-1}(T(v)) = \mathbf{v}$$

- In $\mathbb{R}^n$ if T^{-1} exists, then its matrix satisfies:

$$T^{-1}(T(v)) = A_{T^{-1}} A_T \mathbf{v} = I\mathbf{v}$$

that is, T^{-1} exists iff $(A_T)^{-1}$ exists and $A_{T^{-1}} = (A_T)^{-1}$
(recall that if $BA = I$ then $B = A^{-1}$)

- In $\mathbb{R}^2$ for rotations:

$$A_{T^{-1}} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Example

Is there an inverse to $T : \mathbb{R}^3 \to \mathbb{R}^3$?

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x-y \\ x+2y-3z \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -3 \end{bmatrix}$$

Since $\det(A) = 9$ then the matrix is invertible, and T^{-1} is given by the matrix:

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 5 & 1 \\ 3 & -4 & 1 \\ 3 & -1 & -2 \end{bmatrix} \qquad T^{-1}\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3}u + \frac{5}{9}v + \frac{1}{9}w \\ \frac{1}{3}u - \frac{4}{9}v + \frac{1}{9}w \\ \frac{1}{3}u + \frac{1}{9}v - \frac{2}{9}w \end{bmatrix}$$

Change of Basis for a Linear Transformation

We saw how to find A for a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ using standard basis in both $\mathbb{R}^n$ and $\mathbb{R}^m$.
Now: is there a matrix that represents T wrt two arbitrary bases B and B'?

Theorem

*Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_m\}$ be bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.
Then for all $\mathbf{x} \in \mathbb{R}^n$,*

$$[T(\mathbf{x})]_{B'} = M[\mathbf{x}]_B$$

where $M = A_{[B,B']}$ is the $m \times n$ matrix with the ith column equal to $[T(\mathbf{v}_i)]_{B'}$, the coordinate vector of $T(\mathbf{v}_i)$ wrt the basis B'.

Proof:

$$\begin{array}{cl}
\text{change from } B \text{ to standard} & \mathbf{x} = P_B^{n\times n}[\mathbf{x}]_B \quad \forall \mathbf{x} \in \mathbb{R}^n \\
\downarrow & \\
\text{perform linear transformation} & T(\mathbf{x}) = A\mathbf{x} = AP_B^{n\times n}[\mathbf{x}]_B \\
\text{in standard coordinates} & \\
\downarrow & \\
\text{change to basis } B' & [\mathbf{u}]_{B'} = (P_{B'}^{m\times m})^{-1}\mathbf{u} \quad \forall \mathbf{u} \in \mathbb{R}^m \\
 & [T(\mathbf{x})]_{B'} = (P_{B'}^{m\times m})^{-1}AP_B^{n\times n}[\mathbf{x}]_B \\
 & M = (P_{B'}^{m\times m})^{-1}AP_B^{n\times n}
\end{array}$$

How is M done?

- $P_B = [\mathbf{v}_1\ \mathbf{v}_2\ \dots\ \mathbf{v}_n]$
- $AP_B = A[\mathbf{v}_1\ \mathbf{v}_2\ \dots\ \mathbf{v}_n] = [A\mathbf{v}_1\ A\mathbf{v}_2\ \dots\ A\mathbf{v}_n]$
- $A\mathbf{v}_i = T(\mathbf{v}_i)$: $AP_B = [T(\mathbf{v}_1)\ T(\mathbf{v}_2)\ \dots\ T(\mathbf{v}_n)]$
- $M = P_{B'}^{-1}AP_B = [P_{B'}^{-1}T(\mathbf{v}_1) \quad P_{B'}^{-1}T(\mathbf{v}_2) \quad \dots \quad P_{B'}^{-1}T(\mathbf{v}_n)]$
- $M = [[T(\mathbf{v}_1)]_{B'} \quad [T(\mathbf{v}_2)]_{B'} \quad \dots \quad [T(\mathbf{v}_n)]_{B'}]$

Hence, if we change the basis from the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ the matrix representation of T changes

Similarity

Particular case $m = n$:

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis $\mathbb{R}^n$. Let A be the matrix corresponding to T in standard coordinates: $T(\mathbf{x}) = A\mathbf{x}$. Let

$$P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

be the matrix whose columns are the vectors of B. Then for all $\mathbf{x} \in \mathbb{R}^n$,

$$[T(\mathbf{x})]_B = P^{-1}AP[\mathbf{x}]_B$$

Or, the matrix $A_{[B,B]} = P^{-1}AP$ performs the same linear transformation as the matrix A but expressed it in terms of the basis B.

Similarity

Definition

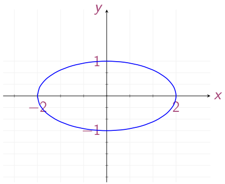
A square matrix C is similar (represent the same linear transformation) to the matrix A if there is an invertible matrix P such that

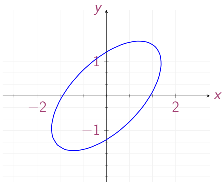
$$C = P^{-1}AP.$$

Similarity defines an equivalence relation:

- (reflexive) a matrix A is similar to itself
- (symmetric) if C is similar to A, then A is similar to C
  $C = P^{-1}AP, \quad A = Q^{-1}CQ, \quad Q = P^{-1}$
- (transitive) if D is similar to C, and C to A, then D is similar to A

Example





- $x^2 + y^2 = 1$ circle in standard form
- $x^2 + 4y^2 = 1$ ellipse in standard form
- $5x^2 + 5y^2 - 6xy = 2$??? Try rotating $\pi/4$ anticlockwise

$$A_T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = P$$

$$\mathbf{v} = P[\mathbf{v}]_B \iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$X^2 + 4Y^2 = 1$$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 3y \\ -x + 5y \end{bmatrix}$$

What is its effect on the xy-plane?
Let's change the basis to

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Find the matrix of T in this basis:

- $C = P^{-1}AP$, A matrix of T in standard basis, P is transition matrix from B to standard

$$C = P^{-1}AP = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Example (cntd)

- the B coordinates of the B basis vectors are

$$[\mathbf{v}_1]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_B, \quad [\mathbf{v}_2]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_B$$

- so in B coordinates T is a stretch in the direction $\mathbf{v}_1$ by 4 and in dir. $\mathbf{v}_2$ by 2:

$$[T(\mathbf{v}_1)]_B = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}_B = 4[\mathbf{v}_1]_B$$

- The effect of T is however the same no matter what basis, only the matrices change! So also in the standard coordinates we must have:

$$A\mathbf{v}_1 = 4\mathbf{v}_1 \qquad A\mathbf{v}_2 = 2\mathbf{v}_2$$

Summary

- Linear transformations and proofs that a given mapping is linear
- two-way relationship between matrices and linear transformations
- Matrix representation of a transformation with respect to two arbitrary basis
- Similarity of square matrices