DM841

Constraint Programming

Compendium

Basic Concepts in Algorithmics

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Outline

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Graphs

Graphs are combinatorial structures useful to model several applications

Terminology:

- $G = (V, E)$, $E \subseteq V \times V$, vertices, edges, $n = |V|, m = |E|$, undirected graphs, subgraph, induced subgraph
- $e = (u, v) \in E$, e incident on u and v; u, v adjacent, edge weight or cost
- particular cases often omitted: self-loops, multiple parallel edges
- degree, δ, Δ, outdegree, indegree
- path $P =< v_0, v_1, \ldots, v_k >$, $(v_0, v_1) \in E, \ldots, (v_{k-1}, v_k) \in E$, $< v_0, v_1 >$ has length 2, $< v_0, v_1, v_2, v_0 >$ cycle, walk, path
- arcs, directed acyclic graph
- digraph strongly connected ($\forall u, v \ \exists (uv)$-path), strongly connected components
- G is a tree ($\implies \exists$ path between any two vertices) $\iff G$ is connected and has $n - 1$ edges $\iff G$ is connected and contains no cycles.
- parent, children, sibling, height, depth

Representing Graphs

Operations:

- Access associated information (NodeArray, EdgeArray, Hashes)
- Navigation: access outgoing edges
- Edge queries: given u and v is there an edge?
- Update: add remove edges, vertices

Data Structures:

- Edge sequences
- Adjacency arrays
- Adjacency lists
- Adjacency matrix

How to choose?

- it depends on the graphs and the application
- if time and space not crucial no need to customize the structures
- use interfaces that make easy to change the data structure
- libraries offer different choices (Boost, lemon, Java `jdsl.graph`)

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Motivations

Questions:

1. How good is the algorithm designed?
2. How hard, computationally, is a given a problem to solve using the most efficient algorithm for that problem?

1. Asymptotic notation, running time bounds
   Approximation theory

2. Complexity theory

Asymptotic notation

$n \in \mathbf{N}$ problem instance size; $\pi \in \Pi_n$ instance π belonging to class Π_n

max time	worst case	$T(n) = \max\{T(\pi) \,:\, \pi \in \Pi_n\}$
average time	average case	$T(n) = \frac{1}{\lvert\Pi_n\rvert}\{\sum_\pi T(\pi) \,:\, \pi \in \Pi_n\}$
min time	best case	$T(n) = \min\{T(\pi) \,:\, \pi \in \Pi_n\}$

Growth rate or asymptotic analysis

$f(n)$ and $g(n)$ same growth rate if	$c \leq \frac{f(n)}{g(n)} \leq d$ for n large
$f(n)$ grows faster than $g(n)$ if	$f(n) \geq c \cdot g(n)$ for all c and n large

big O	$O(f) = \{g(n) \,:\, \exists c > 0, \forall n > n_0 \,:\, g(n) \leq c \cdot f(n)\}$
big omega	$\Omega(f) = \{g(n) \,:\, \exists c > 0, \forall n > n_0 \,:\, g(n) \geq c \cdot f(n)\}$
theta	$\Theta(f) = O(f) \cap \Omega(f)$
(little o	$o(f) = \{g \,:\, g$ grows strictly more slowly than $f\})$

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Machine model

For asymptotic analysis we use RAM machine

- sequential, single processor unit
- all memory access take same amount of time

It is an abstraction from machine architecture: it ignores caches, memories hierarchies, parallel processing (SIMD, multi-threading), etc.

Total execution of a program = total number of instructions executed

We are not interested in constant and lower order terms

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Pseudo-code

We express algorithms in natural language and mathematical notation, and in pseudo-code, which is an abstraction from programming languages C, C++, Java, etc.

(In implementation you can choose your favorite language)

Programs must be correct.
Certifying algorithm: computes a certificate for a post condition (without increasing asymptotic running time)

Good Algorithms

We say that an algorithm A is

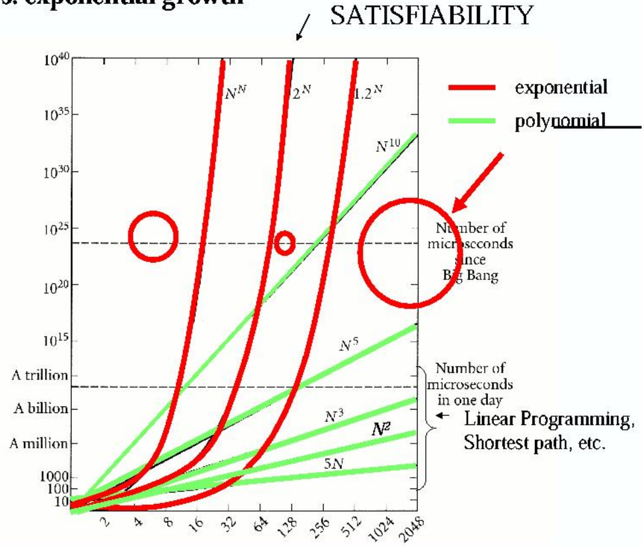
Efficient = good = polynomial time = polytime
iff
there exists polynomial $p(n)$ such that $T(A) = O(p(n))$

There are problems for which no polytime algorithm is known.

Complexity theory classifies problems

Polynomial vs. exponential growth

(Harel 2000)



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Complexity Classes

[Garey and Johnson, 1979]

Consider a Decision Search Problem Π:

- Π is in P if $\exists$ algorithm $\mathcal{A}$ that finds a solution in polynomial time.
- Π is in NP if $\exists$ verification algorithm $\mathcal{A}$ that verifies whether a binary certificate is a solution to the problem in polynomial time.
- a search problem Π' is (polynomially) reducible to Π ($\Pi' \longrightarrow \Pi$) if there exists an algorithm $\mathcal{A}$ that solves Π' by using a hypothetical subroutine $\mathcal{S}$ for Π and except for $\mathcal{S}$ everything runs in polynomial time.
- Π is NP-complete if
  1. it is in NP
  2. there exists some NP-complete problem Π' that reduces to Π ($\Pi' \longrightarrow \Pi$)
- If Π satisfies property 2, but not necessarily property 1, we say that it is NP-hard:

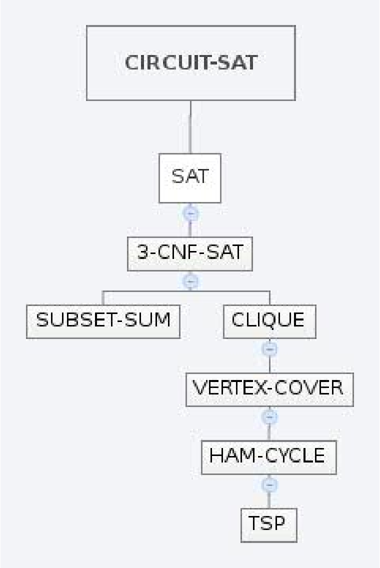
- NP: Class of problems that can be solved in polynomial time by a non-deterministic machine.

  Note: non-deterministic $\neq$ randomized;
  non-deterministic machines are idealized models of computation that have the ability to make perfect guesses.

- NP-complete: Among the most difficult problems in NP; believed to have at least exponential time-complexity for any realistic machine or programming model.

- NP-hard: At least as difficult as the most difficult problems in NP, but possibly not in NP-complete (*i.e.*, may have even worse complexity than NP-complete problems).

NP-Completeness Proofs



Many combinatorial problems are hard
but some problems can be solved efficiently

- Longest path problem is NP-hard
  but not shortest path problem

- SAT for 3-CNF is NP-complete
  but not 2-CNF (linear time algorithm)

- Hamiltonian path is NP-complete
  but not the Eulerian path problem

- TSP on Euclidean instances is NP-hard
  but not where all vertices lie on a circle.

An online compendium on the computational complexity of optimization problems: https://www.csc.kth.se/tcs/compendium/

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Theoretical Analysis

- Worst-case analysis (runtime and quality):
  worst performance of algorithms over all possible instances
- Probabilistic analysis (runtime):
  average-case performance over a given probability distribution of instances
- Average-case (runtime):
  overall possible instances for randomized algorithms
- Asymptotic convergence results (quality)
- Approximation of optimal solutions:
  sometimes possible in polynomial time (*e.g.*, Euclidean TSP),
  but in many cases also intractable (*e.g.*, general TSP);
- Domination
- Algorithm invariance

Approximation Algorithms

Definition: Approximation Algorithms

An algorithm $\mathcal{A}$ is said to be a δ-approximation algorithm if it runs in polynomial time and for every problem instance π with optimal solution value $OPT(\pi)$

$$\text{minimization:} \quad \frac{\mathcal{A}(\pi)}{OPT(\pi)} \leq \delta \quad \delta \geq 1$$

$$\text{maximization:} \quad \frac{\mathcal{A}(\pi)}{OPT(\pi)} \geq \delta \quad \delta \leq 1$$

(δ is called *worst case bound*, *worst case performance*, *approximation factor*, *approximation ratio*, *performance bound*, *performance ratio*, *error ratio*)

Approximation Algorithms

Definition: Polynomial approximation scheme

A family of approximation algorithms for a problem Π, $\{\mathcal{A}_\epsilon\}_\epsilon$, is called a polynomial approximation scheme (PAS), if algorithm $\mathcal{A}_\epsilon$ is a $(1+\epsilon)$-approximation algorithm and its running time is polynomial in the size of the input for each fixed ϵ

Definition: Fully polynomial approximation scheme

A family of approximation algorithms for a problem Π, $\{\mathcal{A}_\epsilon\}_\epsilon$, is called a fully polynomial approximation scheme (FPAS), if algorithm $\mathcal{A}_\epsilon$ is a $(1+\epsilon)$-approximation algorithm and its running time is polynomial in the size of the input and $1/\epsilon$

Useful Graph Algorithms

- Breadth first, depth first search, traversal
- Transitive closure
- Topological sorting
- (Strongly) connected components
- Shortest Path
- Minimum Spanning Tree
- Matching

Randomized Algorithms

Most often algorithms are randomized. Why?

- possibility of gains from re-runs
- adversary argument
- structural simplicity for comparable average performance,
- speed up,
- avoiding loops in the search
- ...

Randomized Algorithms

Definition: Randomized Algorithms

Their running time depends on the random choices made.
Hence, the running time is a random variable.

Las Vegas algorithm: it always gives the correct result but in random runtime (with finite expected value).

Monte Carlo algorithm: the result is not guaranteed correct. Typically halted due to bouned resources.